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Certain generating matrix functions of Charlier matrix polynomials using Weisner's group theoretic method

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Abstract
The present paper discusses a study of a class of Charlier matrix polynomials (CMPs) and its generalized analogue. Certain generating matrix functions, recurrence matrix relations, matrix differential equation, summation formulas and many new results have been discussed for these matrix polynomials. Weisner's group theoretic method is used to obtain matrix generating relations for Charlier matrix polynomials and the details of this method were given in this paper. Finally, we will discuss only briefly the procedure followed.

Keywords Charlier matrix polynomials · Generating matrix functions · Charlier matrix differential equations · Lie algebra

Mathematics Subject Classification 22E60 · 33C45 · 33C65 · 15A60

1 Introduction and motivation

Carl Charlier [2, 16] introduced the family of scalar Charlier polynomials. Special functions are used as an analytical foundation for the majority of problems in mathematical physics it has been solved exactly and finds broad practical applications. With the advancement of knowledge in the field of special functions, it has been the approach of past years of research work to search for new and easy approaches to establish new results and to give easy methods to obtain certain already unknown, known and new results. This approach is better known as the Lie theoretic method. The first significant advancement in this direction was made by Weisner [11–13, 26–28] who exhibits the group theoretic significance of generating functions for hypergeometric, Hermite, and Bessel functions. Then Srivastava and Manocha [25], Manocha [9], Manocha and Jain [3], Miller [14, 15] and McBride [10] present Weisner's method in a systematic manner and thereby laid its firm foundation. Recently the matrix analogues of Laguerre and modified Laguerre matrix polynomials have been considered in [6–8, 19, 20] for matrices in $\mathbb{C}^{N\times N}$ in a number of previous papers, see for example [1, 17, 18] and his work references therein [21–24].

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Motivated by the important works mentioned above, in the present paper mainly the group theoretic approach of Weisner who has been used Weisner devised a method for obtaining generating matrix relations for Charlier matrix polynomials, which satisfy certain conditions. In Sect. 2, we give the definition, many new results and known properties of Charlier matrix polynomials. In Sect. 3, we use the representation theory of the Lie algebra method and derive certain generating matrix relations involving these matrix polynomials by using Weisner's group theoretic techniques.

1.1 Preliminaries

In this subsection, we review some useful definitions, lemmas and the concepts related to the matrix functional calculus. Throughout this paper, for a matrix A in $\mathbb{C}^{N\times N}$, its spectrum is denoted by $\sigma(A)$ where $\sigma(A)$ is the set of the eigenvalues of A. The matrices I and $\mathbf{0}$ will denote the identity matrix and null matrix in $\mathbb{C}^{N\times N}$, respectively.

Definition 1.1 [4, 5] For a matrix $A \in \mathbb{C}^{N\times N}$ such that $\sigma(A)$ does not contain 0 or a negative integer ($\sigma(A)\cap\mathbb{Z}^- = \emptyset$ where $\emptyset$ is an empty set), the matrix form of the shifted factorial is defined by

$$(A)_n = \begin{cases} A(A+I)\dots(A+(n-1)I) = \Gamma(A+nI)\Gamma^{-1}(A), & n \geq 1; \\ I, & n = 0, \end{cases} \tag{1.1}$$

where $\Gamma(A)$ is an invertible matrix in $\mathbb{C}^{N\times N}$ and $\Gamma^{-1}(A)$ is inverse Gamma matrix function.

For A is an arbitrary matrix of $\mathbb{C}^{N\times N}$ and using (1.1), we have the relations

$$\begin{aligned} &(A)_{n+k} = (A)_n(A+nI)_k = (A)_k(A+kI)_n, \\ &(-nI)_k = \begin{cases} \frac{(-1)^k\, n!}{(n-k)!} I, & 0 \leq k \leq n; \\ \mathbf{0}, & k > n, \end{cases} \end{aligned} \tag{1.2}$$

and

$$\Gamma(I-A-nI)\Gamma^{-1}(I-A) = (-1)^n[(A)_n]^{-1}, \tag{1.3}$$

where $\Gamma(I-A)$ is an invertible matrix. In view of Eqs. (1.1) and (1.3), we observe that

$$\Gamma(A+I)\Gamma^{-1}(A-(k-1)I) = (-1)^k(-A)_k, \tag{1.4}$$

where $\Gamma(A-(k-1)I)$ is an invertible matrix.

2 Charlier matrix polynomials: definition and properties

Let us assume that A is a matrix in $\mathbb{C}^{N\times N}$ whose A satisfy the condition $-k \notin \sigma(A)$, for every integer $k > 0$ ($\sigma(A)$ =spectrum of A), λ is a complex parameter whose real part is a positive and let us consider a new generating matrix function which represents the Charlier matrix polynomials $\mathbf{C}_n(A;\lambda,a;x)$ by

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\mathbf{C}_n(A;\lambda,a;x) = a^t\left(1-\frac{t}{\lambda x}\right)^A,\ x > 0,\ \left|\frac{t}{\lambda x}\right| < 1,\ \lambda x \neq 0,\ a > 0,\ a \neq 1. \tag{2.1}$$

From (2.1), we obtain a series representation of Charlier matrix polynomials of degree n as

$$\mathbf{C}_n(A;\lambda,a;x)=n!\sum_{k=0}^{n}\frac{(-1)^k}{k!(n-k)!}\Gamma(A+I)\Gamma^{-1}(A-(k-1)I)(\log(a))^{n-k}(\lambda x)^{-k}. \tag{2.2}$$

Theorem 2.1 *The CMPs is defined by*

$$\mathbf{C}_n(A;\lambda,a;x)=(\log(a))^n\ {}_2F_0\Big(-nI,-A;-;-\frac{1}{\lambda x\log(a)}\Big),\ \lambda x\log(a)\neq 0. \tag{2.3}$$

Proof Using the relations (1.2) and (1.4) in (2.2), we get

$$\mathbf{C}_n(A;\lambda,a;x)=\sum_{k=0}^{n}\frac{1}{k!}(-nI)_k(-1)^k(-A)_k(\log(a))^{n-k}(\lambda x)^{-k},$$

the representation (2.3) is obtained. □

Theorem 2.2 *The generating matrix function for CMPs is*

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{t^n}{n!}(B)_n\mathbf{C}_n(A;\lambda,a;x)&=\Big(1-t\log(a)\Big)^{-B}{}_2F_0\Big(B,-A;-;\frac{t}{\lambda x(1-t\log(a))}\Big),\\ |t\log(a)|&<1,\end{aligned} \tag{2.4}$$

where B is a matrix in $\mathbb{C}^{N\times N}$.

Proof By using the series representation of CMPs given in (2.2) and (1.2), we get

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{t^n}{n!}(B)_n\mathbf{C}_n(A;\lambda,a;x)&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k}{k!(n-k)!}(B)_n(-1)^k(-A)_k(\log(a))^{n-k}(\lambda x)^{-k}t^n\\ &=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{k!n!}(B)_k(B+kI)_n(-A)_k(\log(a))^n(\lambda x)^{-k}t^{n+k}\\ .&=\sum_{k=0}^{\infty}\frac{1}{k!}(B)_k(-A)_k(\log(a))^n(\lambda x)^{-k}(1-t\log(a))^{-(B+kI)}t^k\\ &=\Big(1-t\log(a)\Big)^{-B}\sum_{k=0}^{\infty}\frac{1}{k!}(B)_k(-A)_k(\lambda x)^{-k}\Big(\frac{t}{1-t\log(a)}\Big)^k,\end{aligned}$$

which gives the desired result. □

The recurrence matrix relations are also presented in the next theorem:

Theorem 2.3 *For the Charlier matrix polynomials, the matrix recurrence relations*

$$x\frac{d}{dx}\mathbf{C}_n(A;\lambda,a;x)+n\mathbf{C}_n(A;\lambda,a;x)=n\log(a)\mathbf{C}_{n-1}(A;\lambda,a;x),\ n\geq 1, \tag{2.5}$$

$$x\frac{d}{dx}\mathbf{C}_n(A;\lambda,a;x)+(A-\lambda x\log(a)I)\mathbf{C}_n(A;\lambda,a;x)=-\lambda x\mathbf{C}_{n+1}(A;\lambda,a;x) \tag{2.6}$$

and

$$\begin{aligned}&\lambda x\mathbf{C}_{n+1}(A;\lambda,a;x)+n\log(a)\mathbf{C}_{n-1}(A;\lambda,a;x)\\ &\quad+(A-(n+\lambda x\log(a))I)\mathbf{C}_n(A;\lambda,a;x)=\mathbf{0},\ n\geq 1,\end{aligned} \tag{2.7}$$

hold.

Proof Differentiating (2.1) with respect to x, we can write

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\frac{d}{dx}\mathbf{C}_n(A;\lambda,a;x)=\frac{t}{\lambda x^2}Aa^t\left(1-\frac{t}{\lambda x}\right)^{A-I},\ \left|\frac{t}{\lambda x}\right|<1. \tag{2.8}$$

Multiplying Eq. (2.8) by $\left(1-\frac{t}{\lambda x}\right)$ and using (2.1), we get

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{t^n}{n!}\frac{d}{dx}\mathbf{C}_n(A;\lambda,a;x)-\frac{1}{\lambda x}\sum_{n=0}^{\infty}\frac{t^{n+1}}{n!}\frac{d}{dx}\mathbf{C}_n(A;\lambda,a;x)\\&=\frac{A}{\lambda x^2}\sum_{n=0}^{\infty}\frac{t^{n+1}}{n!}\mathbf{C}_n(A;\lambda,a;x).\end{aligned}$$

Comparing the coefficient of t^n yields the matrix differential recurrence relation

$$\lambda x^2\frac{d}{dx}\mathbf{C}_n(A;\lambda,a;x)-nx\frac{d}{dx}\mathbf{C}_{n-1}(A;\lambda,a;x)=nA\mathbf{C}_{n-1}(A;\lambda,a;x). \tag{2.9}$$

On the other hand, differentiating (2.1) with respect to t, we have

$$\sum_{n=0}^{\infty}\frac{nt^{n-1}}{n!}\mathbf{C}_n(A;\lambda,a;x)=-\frac{A}{\lambda x}a^t\left(1-\frac{t}{\lambda x}\right)^{A-I}+\log(a)a^t\left(1-\frac{t}{\lambda x}\right)^{A}. \tag{2.10}$$

Using (2.8) into (2.10) and subtracting, we get

$$\sum_{n=0}^{\infty}\frac{nt^n}{n!}\mathbf{C}_n(A;\lambda,a;x)=-x\sum_{n=0}^{\infty}\frac{t^n}{n!}\frac{d}{dx}\mathbf{C}_n(A;\lambda,a;x)+\log(a)\sum_{n=0}^{\infty}\frac{t^{n+1}}{n!}\mathbf{C}_n(A;\lambda,a;x).$$

Comparing the coefficient of t^n, we obtain (2.5).

Multiplying Eq. (2.5) by λx, using (2.9) and subtracting $\frac{d}{dx}\mathbf{C}_n(A;\lambda,a;x)$, we have

$$x\frac{d}{dx}\mathbf{C}_{n-1}(A;\lambda,a;x)+(A-\lambda x\log(a))\mathbf{C}_{n-1}(A;\lambda,a;x)=-\lambda xA\mathbf{C}_n(A;\lambda,a;x). \tag{2.11}$$

Replacing n by $n+1$ in (2.11), we obtain (2.6). From (2.5) and (2.6), we derive the desired pure matrix recurrence relation (2.7). □

Theorem 2.4 *The finite summation for CMPs is*

$$\left(1-\frac{y}{x}\right)^n\mathbf{C}_n(A;\lambda,a;x-y)=\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\left(-\frac{y\log(a)}{x}\right)^k\mathbf{C}_{n-k}(A;\lambda,a;x),$$
$$\left|\frac{y}{x}\right|<1,x\neq 0. \tag{2.12}$$

Proof In (2.1), taking $u=\frac{t}{\lambda x}$, we write the generating matrix relation

$$\sum_{n=0}^{\infty}\frac{u^n}{n!}(\lambda x)^n\mathbf{C}_n(A;\lambda,a;x)=a^{\lambda xu}\left(1-u\right)^{A},\ x>0,|u|<1. \tag{2.13}$$

Let us consider the following equation:

$$\sum_{n=0}^{\infty}\frac{u^n}{n!}(\lambda x)^n\mathbf{C}_n(A;\lambda,a;x)=a^{\lambda yu}\sum_{n=0}^{\infty}\frac{u^n}{n!}(\lambda(x-y))^n\mathbf{C}_n(A;\lambda,a;x-y).$$

Multiplying the both sides of this equation by $a^{-\lambda yu}$, replacing n by $n-k$ and using (2.13), we give

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{u^n}{n!}(\lambda(x-y))^n\mathbf{C}_n(A;\lambda,a;x-y)&=a^{-\lambda yu}\sum_{n=0}^{\infty}\frac{u^n}{n!}(\lambda x)^n\mathbf{C}_n(A;\lambda,a;x)\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{u^{n+k}}{k!n!}(-\lambda y\log(a))^k(\lambda x)^n\mathbf{C}_n(A;\lambda,a;x)\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{u^n}{k!(n-k)!}(-\lambda y\log(a))^k(\lambda x)^{n-k}\mathbf{C}_{n-k}(A;\lambda,a;x).\end{aligned}$$

Comparing the coefficient of t^n, the relation is established. □

Corollary 2.1 *The CMPs satisfy*

$$\big(1+t\big)^n\mathbf{C}_n(A;\lambda,a;x(1+t))=\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\Big(t\log(a)\Big)^k\mathbf{C}_{n-k}(A;\lambda,a;x),\ |t|<1. \tag{2.14}$$

Proof Taking $-\frac{y}{x}\longrightarrow t$ in Eq. (2.12), we have the desired result. □

Theorem 2.5 *The generating matrix relation for CMPs is*

$$\sum_{k=0}^{\infty}\frac{1}{k!}\mathbf{C}_{n+k}(A;\lambda,a;x)t^k=a^t\Big(1-\frac{t}{\lambda x}\Big)^A\mathbf{C}_n\Big(A;\lambda,a;x\big(1-\frac{t}{\lambda x}\big)\Big);\ \Big|\frac{t}{\lambda x}\Big|<1. \tag{2.15}$$

Proof If we replace t by $t+v$ in (2.1), we get

$$a^{t+v}\Big(1-\frac{t+v}{\lambda x}\Big)^A=\sum_{n=0}^{\infty}\frac{(t+v)^n}{n!}\mathbf{C}_n(A;\lambda,a;x). \tag{2.16}$$

By expanding the binomial theorem $(t+v)^n$, replacing n by $n+k$ and simplifying, we can write

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(t+v)^n}{n!}\mathbf{C}_n(A;\lambda,a;x)&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{k!(n-k)!}\mathbf{C}_{n+k}(A;\lambda,a;x)t^kv^n\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{n!k!}\mathbf{C}_{n+k}(A;\lambda,a;x)t^kv^n.\end{aligned}$$

If we associate the factors depending on v in the left member of (2.16), we get

$$a^{t+v}\Big(1-\frac{t+v}{\lambda x}\Big)^A=a^t\Big(1-\frac{t}{\lambda x}\Big)^Aa^v\Big(1-\frac{v}{\lambda x\big(1-\frac{t}{\lambda x}\big)}\Big)^A=\sum_{n=0}^{\infty}\frac{(t+v)^n}{n!}\mathbf{C}_n(A;\lambda,a;x).$$

By means of (2.1), we can write

$$a^{v}\Big(1-\frac{v}{\lambda x\big(1-\frac{t}{\lambda x}\big)}\Big)^{A}=\sum_{n=0}^{\infty}\frac{v^{n}}{n!}\mathbf{C}_{n}\Big(A;\lambda,a;x\big(1-\frac{t}{\lambda x}\big)\Big).$$

Therefore, the left member of (2.16) has the expansion in powers of v:

$$a^{t+v}\Big(1-\frac{t+v}{\lambda x}\Big)^{A}=a^{t}\Big(1-\frac{t}{\lambda x}\Big)^{A}\sum_{n=0}^{\infty}\frac{v^{n}}{n!}\mathbf{C}_{n}\Big(A;\lambda,a;x\big(1-\frac{t}{\lambda x}\big)\Big).$$

Hence

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{n!k!}\mathbf{C}_{n+k}(A;\lambda,a;x)t^{k}v^{n}=a^{t}\Big(1-\frac{t}{\lambda x}\Big)^{A}\sum_{n=0}^{\infty}\frac{v^{n}}{n!}\mathbf{C}_{n}\Big(A;\lambda,a;x\big(1-\frac{t}{\lambda x}\big)\Big).$$

Comparing the coefficients of v^n in the above equation, we get the assertion (2.15) of Theorem 2.5. □

Theorem 2.6 *The CMPs is a solution of second order matrix differential equation*

$$\Big[x^{2}D^{2}I+((1+n-\lambda x\log(a))I+A)xD+nA\Big]\mathbf{C}_{n}(A;\lambda,a;x)=\boldsymbol{0}. \tag{2.17}$$

Proof Starting from (2.5) and (2.9), we get

$$\Big(A-\lambda x\log(a)I\Big)xD\mathbf{C}_{n}(A;\lambda,a;x)+nA\mathbf{C}_{n}(A;\lambda,a;x)=-nx\log(a)D\mathbf{C}_{n-1}(A;\lambda,a;x). \tag{2.18}$$

Differentiating (2.5) with respect to x and multiplying by x, we get

$$x^{2}D^{2}\mathbf{C}_{n}(A;\lambda,a;x)+(n+1)xD\mathbf{C}_{n}(A;\lambda,a;x)=nx\log(a)D\mathbf{C}_{n-1}(A;\lambda,a;x). \tag{2.19}$$

Adding the result to (2.18) and (2.19) gives the second order matrix differential equation for CMPs. □

Starting from (2.2), we get the another formula of matrix recurrence relation for the CMPs

$$\mathbf{C}_{n+1}(A;\lambda,a;x)=\log(a)\mathbf{C}_{n}(A;\lambda,a;x)-\frac{A}{\lambda x}\mathbf{C}_{n}(A-I;\lambda,a;x),\ \mathbf{C}_{0}(A;\lambda,a;x)=I, \tag{2.20}$$

and if we take the backward shift operator $\mathbb{E}f(A)=f(A-I)$ as

$$\begin{aligned}\mathbf{C}_{n}(A;\lambda,a;x)&=\Big(\log(a)I-\frac{1}{\lambda x}A\mathbb{E}\Big)\mathbf{C}_{n-1}(A;\lambda,a;x)\\&=\Big(I-\frac{1}{\lambda x\log(a)}A\mathbb{E}\Big)\log(a)\mathbf{C}_{n-1}(A;\lambda,a;x),\end{aligned}$$

which gives (2.20) and we obtain the new explicit representation for CMPs

$$\mathbf{C}_{n}(A;\lambda,a;x)=\Big(I-\frac{1}{\lambda x\log(a)}A\mathbb{E}\Big)^{n}\Big(\log(a)\Big)^{n}. \tag{2.21}$$

Expanding the right hand side of the previous expression, using the binomial expansion, yields the new explicit representation for CMPs

$$\begin{aligned}\mathbf{C}_n(A;\lambda,a;x) &= \sum_{k=0}^{n}\frac{(-1)^k n!}{k!(n-k)!}\left(\frac{A\mathbb{E}}{\lambda x\log(a)}\right)^k\left(\log(a)\right)^n\\ &= \sum_{k=0}^{n}\frac{(-1)^k n!}{k!(n-k)!}\left(\frac{\mathbb{Q}^{(k)}}{(\lambda x\log(a))^k}\right)\left(\log(a)\right)^n,\end{aligned}\tag{2.22}$$

where $\mathbb{Q}^{(0)}=I$, $\mathbb{Q}^{(1)}=A\mathbb{E}1=A$, $\mathbb{Q}^{(2)}=(A\mathbb{E})^2 1=(A\mathbb{E})(A\mathbb{E})1=(A\mathbb{E})A=A(A-I)$ and general $\mathbb{Q}^{(k)}=(A\mathbb{E})^k 1=A(A-I)\ldots(A-(k-1)I)$.

Thus, we list the first elements of the Chartier matrix polynomials

$$\mathbf{C}_0(A;\lambda,a;x)=I=\mathbb{Q}^{(0)},$$

$$\mathbf{C}_1(A;\lambda,a;x)=\frac{A}{\lambda x}-\log(a)I=\left(\frac{A}{\lambda x\log(a)}-I\right)\log(a)=\left(\frac{\mathbb{Q}^{(1)}}{\lambda x\log(a)}-\mathbb{Q}^{(0)}\right)\log(a),$$

$$\begin{aligned}\mathbf{C}_2(A;\lambda,a;x)&=\left(\frac{A(A-I)}{(\lambda x\log(a))^2}-2\frac{A}{\lambda x\log(a)}+I\right)\left(\log(a)\right)^2\\ &=\left(\frac{\mathbb{Q}^{(2)}}{(\lambda x\log(a))^2}-2\frac{\mathbb{Q}^{(1)}}{\lambda x\log(a)}+\mathbb{Q}^{(0)}\right)\left(\log(a)\right)^2,\end{aligned}$$

$$\begin{aligned}\mathbf{C}_3(A;\lambda,a;x)&=\left(\frac{A(A-I)(A-2I)}{(\lambda x\log(a))^3}-3\frac{A(A-I)}{(\lambda x\log(a))^2}+3\frac{A}{\lambda x\log(a)}-I\right)\left(\log(a)\right)^3\\ &=\left(\frac{\mathbb{Q}^{(3)}}{(\lambda x\log(a))^3}-3\frac{\mathbb{Q}^{(2)}}{(\lambda x\log(a))^2}+3\frac{\mathbb{Q}^{(1)}}{\lambda x\log(a)}-\mathbb{Q}^{(0)}\right)\left(\log(a)\right)^3.\end{aligned}$$

Summary of these results are given in the next theorem.

Theorem 2.7 *The CMPs satisfy:*

- *For* $n\geq 1$

$$\mathbf{C}_{n+1}(A;\lambda,a;x)=\log(a)\mathbf{C}_n(A;\lambda,a;x)-\frac{A}{\lambda x}\mathbf{C}_n(A-I;\lambda,a;x),\ \mathbf{C}_0(A;\lambda,a;x)=I.$$

- *In* (2.21) *and* (2.22) *are defined the matrix versions of new explicit representations for CMPs.*

3 Linear differential operators

In accordance with the discussion in the preceding section, the partial differential matrix equation are constructed from (2.17) by replacing n by $y\frac{\partial}{\partial y}$, D by $\frac{\partial}{\partial x}$ and $\mathbf{C}_n(A;\lambda,a;x)$ by $\mathbf{C}_n(A;\lambda,a;x,y)=y^n\mathbf{C}_n(A;\lambda,a;x)$:

$$\left[x^2\frac{\partial^2}{\partial x^2}I+\left((1+y\frac{\partial}{\partial y}-\lambda x\log(a))I+A\right)x\frac{\partial}{\partial x}+Ay\frac{\partial}{\partial y}\right]\mathbf{C}_n(A;\lambda,a;x,y)=\mathbf{0}.\tag{3.1}$$

We rewrite (3.1) in the form

$$\left[x^2\frac{\partial^2}{\partial x^2}I+\left((1-\lambda x\log(a))I+A\right)x\frac{\partial}{\partial x}+xy\frac{\partial}{\partial y\partial x}I+Ay\frac{\partial}{\partial y}\right]\mathbf{C}_n(A;\lambda,a;x,y)=\mathbf{0},\tag{3.2}$$

where

$$\mathbb{L} = \mathbb{L}\left(x, \frac{\partial}{\partial x}, y\frac{\partial}{\partial y}\right) = x^2\frac{\partial^2}{\partial x^2}I + \left((1-\lambda x\log(a))I + A\right)x\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y\partial x}I + Ay\frac{\partial}{\partial y}.$$

Since

$$\mathbb{L}\left(x, \frac{\partial}{\partial x}, y\frac{\partial}{\partial y}\right)[\mathbf{C}_n(A;\lambda,a;x)y^n] = y^n\mathbb{L}\left(x, \frac{d}{dx}, n\right)[\mathbf{C}_n(A;\lambda,a;x)],$$

for $y \neq 0$, we get

$$\mathbb{L}\left(x, \frac{\partial}{\partial x}, y\frac{\partial}{\partial y}\right)[\mathbf{C}_n(A;\lambda,a;x)y^n] = \mathbf{0},$$

if and only if

$$\mathbb{L}\left(x, \frac{d}{dx}, n\right)[\mathbf{C}_n(A;\lambda,a;x)] = \mathbf{0}.$$

Again using the technique described in detail in Sect. 2, we determine with the aid of (2.5) and (2.6) the first order linear differential operators $\mathbb{I}$, $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ for CMPs such that

$$\mathbb{A}\left[\mathbf{C}_n(A;\lambda,a;x)y^n\right] = n\mathbf{C}_n(A;\lambda,a;x)y^n, \tag{3.3}$$

$$\mathbb{B}\left[\mathbf{C}_n(A;\lambda,a;x)y^n\right] = n\log(a)\mathbf{C}_{n-1}(A;\lambda,a;x)y^{n-1} \tag{3.4}$$

and

$$\mathbb{C}\left[\mathbf{C}_n(A;\lambda,a;x)y^n\right] = -\lambda\mathbf{C}_{n+1}(A;\lambda,a;x)y^{n+1}, \tag{3.5}$$

where

$$\mathbb{A} = y\frac{\partial}{\partial y}I, \tag{3.6}$$

$$\mathbb{B} = \frac{x}{y}\frac{\partial}{\partial x}I + \frac{\partial}{\partial y}I \tag{3.7}$$

and

$$\mathbb{C} = y\frac{\partial}{\partial x}I + y\left(\frac{1}{x}A - \lambda\log(a)I\right). \tag{3.8}$$

Therefore, the operators $\mathbb{I}$, $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ generate a Lie algebra.

Theorem 3.1 *The operators $\mathbb{I}$, $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ satisfy the commutation relations:*

$$(i)\quad [\mathbb{A},\mathbb{B}] = -\mathbb{B}, \qquad (ii)\quad [\mathbb{A},\mathbb{C}] = \mathbb{C}, \qquad (iii)\quad [\mathbb{B},\mathbb{C}] = -\lambda\log(a)\mathbb{I}, \tag{3.9}$$

where $\mathbb{I}$ stands for the identity operator.

Proof We find that

$$\mathbb{A}\mathbb{B}\mathbf{C}_n(A;\lambda,a;x,y) = y\frac{\partial}{\partial y}\left[\frac{x}{y}\frac{\partial}{\partial x}I + \frac{\partial}{\partial y}I\right]\mathbf{C}_n(A;\lambda,a;x,y).$$

Hence, on simplification, we have

$$\begin{aligned}\mathbb{A}\mathbb{B}\mathbf{C}_n(A;\lambda,a;x,y) &= x\frac{\partial^2}{\partial y\partial x}\mathbf{C}_n(A;\lambda,a;x,y) - \frac{x}{y}\frac{\partial}{\partial x}\mathbf{C}_n(A;\lambda,a;x,y)\\ &\quad + y\frac{\partial^2}{\partial y^2}\mathbf{C}_n(A;\lambda,a;x,y).\end{aligned} \tag{3.10}$$

On the other hand, we have

$$\mathbb{B}\mathbb{A}\mathbf{C}_n(A;\lambda,a;x,y)=\left(\frac{x}{y}\frac{\partial}{\partial x}I+\frac{\partial}{\partial y}I\right)y\frac{\partial}{\partial y}\mathbf{C}_n(A;\lambda,a;x,y).$$

This is to be simplified as

$$\begin{aligned}\mathbb{B}\mathbb{A}\mathbf{C}_n(A;\lambda,a;x,y)=&x\frac{\partial^2}{\partial x\partial y}\mathbf{C}_n(A;\lambda,a;x,y)+\frac{\partial}{\partial y}\mathbf{C}_n(A;\lambda,a;x,y)\\&+y\frac{\partial^2}{\partial y^2}\mathbf{C}_n(A;\lambda,a;x,y).\end{aligned}\tag{3.11}$$

Then, for $\frac{\partial^2}{\partial x\partial y}=\frac{\partial^2}{\partial y\partial x}$, subtracting (3.11) from (3.10), we get

$$\begin{aligned}&[\mathbb{A},\mathbb{B}]\mathbf{C}_n(A;\lambda,a;x,y)=\big(\mathbb{A}\mathbb{B}-\mathbb{B}\mathbb{A}\big)\mathbf{C}_n(A;\lambda,a;x,y)\\&\quad=-\frac{x}{y}\frac{\partial}{\partial x}\mathbf{C}_n(A;\lambda,a;x,y)-\frac{\partial}{\partial y}\mathbf{C}_n(A;\lambda,a;x,y)=-\mathbb{B}\mathbf{C}_n(A;\lambda,a;x,y).\end{aligned}$$

Thus, we have the required result $[\mathbb{A},\mathbb{B}]=-\mathbb{B}$. Similarly, we can proceed to calculate each of the following results $[\mathbb{A},\mathbb{C}]$ and $[\mathbb{B},\mathbb{C}]$, which completes the proof theorem. □

Now, we know that the operator $\frac{1}{x}\mathbb{L}$ given by

$$\begin{aligned}\frac{1}{x}\mathbb{L}\mathbf{C}_n(A;\lambda,a;x,y)=&\,x\frac{\partial^2}{\partial x^2}\mathbf{C}_n(A;\lambda,a;x,y)\\&+\Big((1-\lambda x\log(a))I+A\Big)\frac{\partial}{\partial x}\mathbf{C}_n(A;\lambda,a;x,y)\\&+y\frac{\partial}{\partial y\partial x}\mathbf{C}_n(A;\lambda,a;x,y)+\frac{Ay}{x}\frac{\partial}{\partial y}\mathbf{C}_n(A;\lambda,a;x,y).\end{aligned}$$

Also, in making the calculation of $[\mathbb{B},\mathbb{C}]$, we found that

$$\begin{aligned}\mathbb{C}\mathbb{B}\mathbf{C}_n(A;\lambda,a;x,y)=&\,x\frac{\partial^2}{\partial x^2}\mathbf{C}_n(A;\lambda,a;x,y)+(1+A-\lambda x\log(a))\frac{\partial}{\partial x}\mathbf{C}_n(A;\lambda,a;x,y)\\&+y\frac{\partial^2}{\partial x\partial y}\mathbf{C}_n(A;\lambda,a;x,y)+(\frac{A}{x}-\lambda\log(a))y\frac{\partial}{\partial y}\mathbf{C}_n(A;\lambda,a;x,y).\end{aligned}$$

Hence, we get

$$\frac{1}{x}\mathbb{L}\mathbf{C}_n(A;\lambda,a;x,y)-\mathbb{C}\mathbb{B}\mathbf{C}_n(A;\lambda,a;x,y)=\lambda\log(a)y\frac{\partial}{\partial y}\mathbf{C}_n(A;\lambda,a;x,y),$$

or equivalently,

$$\frac{1}{x}\mathbb{L}\mathbf{C}_n(A;\lambda,a;x,y)=\Big[\mathbb{C}\mathbb{B}+\lambda\log(a)\mathbb{A}\Big]\mathbf{C}_n(A;\lambda,a;x,y),$$

Therefore, it can be expressed as:

$$\frac{1}{x}\mathbb{L}=\mathbb{C}\mathbb{B}+\lambda\log(a)\mathbb{A};\,x\neq0.\tag{3.12}$$

Now, we show that

$$\begin{aligned}
&\left[\frac{1}{x}\mathbb{L}, \mathbb{C}\right]\mathbf{C}_n(A;\lambda,a;x,y)\\
&\quad=\left(\frac{1}{x}\mathbb{L}\mathbb{C}-\mathbb{C}\frac{1}{x}\mathbb{L}\right)\mathbf{C}_n(A;\lambda,a;x,y)\\
&\quad=\left(\left(\mathbb{C}\mathbb{B}+\lambda\log(a)\mathbb{A}\right)\mathbb{C}-\mathbb{C}\left(\mathbb{C}\mathbb{B}+\lambda\log(a)\mathbb{A}\right)\right)\mathbf{C}_n(A;\lambda,a;x,y)\\
&\quad=\left(\mathbb{C}\mathbb{B}\mathbb{C}+\lambda\log(a)\mathbb{A}\mathbb{C}-\mathbb{C}\mathbb{C}\mathbb{B}-\mathbb{C}\lambda\log(a)\mathbb{A}\right)\mathbf{C}_n(A;\lambda,a;x,y).
\end{aligned}\tag{3.13}$$

With the help of (3.9), we simplify $\mathbb{A}\mathbb{C}-\mathbb{C}\mathbb{A}$ as follows:

$$\mathbb{A}\mathbb{C}-\mathbb{C}\mathbb{A}=\mathbb{C}.$$

Hence, with the further aid of the commutator relations from (3.9), we get

$$\mathbb{C}\left(\mathbb{B}\mathbb{C}-\mathbb{C}\mathbb{B}\right)=\mathbb{C}\left(-\lambda\log(a)\mathbf{C}_n(A;\lambda,a;x,y)\right)=-\lambda\log(a)\mathbb{C}.$$

From (3.13), we obtain

$$\left[\frac{1}{x}\mathbb{L}, \mathbb{C}\right]=\mathbf{0}.$$

Thus, we establish that the operator $\frac{1}{x}\mathbb{L}$ commutes with $\mathbb{C}$. In the same way, we can calculate each of the following results $\left[\frac{1}{x}\mathbb{L}, \mathbb{A}\right]=\mathbf{0}$ and $\left[\frac{1}{x}\mathbb{L}, \mathbb{B}\right]=\mathbf{0}$. We summarize these results in the following theorem.

Theorem 3.2 *Linear differential operators $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ defined in* (3.5), (3.6) *and* (3.7) *commute with the operator $\frac{1}{x}\mathbb{L}$ as follows*

$$(i)\ \left[\frac{1}{x}\mathbb{L}, \mathbb{A}\right]=\mathbf{0},\qquad (ii)\ \left[\frac{1}{x}\mathbb{L}, \mathbb{B}\right]=\mathbf{0},\qquad (iii)\ \left[\frac{1}{x}\mathbb{L}, \mathbb{C}\right]=\mathbf{0},\tag{3.14}$$

where the operator $\mathbb{L}$ is as defined in (3.2).

We express the extended form of the group generated by the operators $\mathbb{I}$, $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ as follows:

$$e^{a\mathbb{A}}f(x,y,A)=f\left(x,ye^{a},A\right),\tag{3.15}$$

$$e^{b\mathbb{B}}f(x,y,A)=f\left(x\left(1+\frac{b}{y}\right),b+y,A\right),\ y\neq 0\tag{3.16}$$

and

$$e^{c\mathbb{C}}f(x,y,A)=e^{-\lambda cy\log(a)}\left(1+\frac{cy}{x}\right)^{A}f\left(x+cy,y,A\right);\ x\neq 0,\left|\frac{cy}{x}\right|<1,\tag{3.17}$$

where $f(x,y,A)$ is an arbitrary matrix function and a, b and c are arbitrary constants.

From the above relations, we get

$$
\begin{aligned}
&e^{c\mathbb{C}}e^{b\mathbb{B}}e^{a\mathbb{A}}f(x,y,A)\\
&\quad = e^{-\lambda cy\log(a)}\left(1+\frac{cy}{x}\right)^{A} f\left((x+cy)\left(1+\frac{b}{y}\right),(b+y)e^{a},A\right);\, x,y\neq 0,\left|\frac{cy}{x}\right|<1.
\end{aligned} \tag{3.18}
$$

3.1 Generating matrix functions derived from the operators

Here, we have determined the new generating matrix relations between CMPs. For arbitrary constants b and c the operator $e^{c\mathbb{C}}e^{b\mathbb{B}}\frac{1}{x}\mathbb{L}[\mathbf{C}_n(A;\lambda,a;x)y^n]$ will transform solutions of $\mathbb{L}$ into solutions of $\mathbb{L}$; in other words, we have

$$
e^{c\mathbb{C}}e^{b\mathbb{B}}\frac{1}{x}\mathbb{L}\mathbf{C}_n(A;\lambda,a;x,y)=\frac{1}{x}\mathbb{L}e^{c\mathbb{C}}e^{b\mathbb{B}}\mathbf{C}_n(A;\lambda,a;x,y).
$$

Now it follows from (3.3) that $\mathbf{C}_n(A;\lambda,a;x,y)$ is a solution of the system

$$
\mathbb{L}\mathbf{C}_n(A;\lambda,a;x,y)=\mathbf{0} \quad\text{and}\quad (\mathbb{A}-n\mathbb{I})\mathbf{C}_n(A;\lambda,a;x,y)=\mathbf{0},
$$

for the matrix function $\mathbf{C}_n(A;\lambda,a;x,y)=y^n\mathbf{C}_n(A;\lambda,a;x)$. Since $\mathbb{B}$ and $\mathbb{C}$ commute with $\frac{1}{x}\mathbb{L}$, we have

$$
e^{c\mathbb{C}}e^{b\mathbb{B}}\frac{1}{x}\mathbb{L}[\mathbf{C}_n(A;\lambda,a;x)y^n]=\frac{1}{x}\mathbb{L}e^{c\mathbb{C}}e^{b\mathbb{B}}\mathbb{L}[\mathbf{C}_n(A;\lambda,a;x)y^n].
$$

Before discussing particular cases (3.18), it is of interest to mention that the operators $\mathbb{B}$ and $\mathbb{C}$ being commutative. The above generating matrix relations which yield a good number of a particular generating relations by attributing different values to $a=0$ appears to be new. Therefore, we will consider the following three cases of the transformed function $e^{c\mathbb{C}}e^{b\mathbb{B}}\frac{1}{x}\mathbb{L}[\mathbf{C}_n(A;\lambda,a;x)y^n]$:

- Case 1. $b=1, c=0$,
- Case 2. $b=0, c=1$,
- Case 3. $bc\neq 0$.

Case 1. From (3.16), we know that for an arbitrary matrix function

$$
e^{\mathbb{B}}f(x,y,A)=f\left(x\left(1+\frac{1}{y}\right),1+y,A\right),
$$

then it follows that

$$
e^{\mathbb{B}}[\mathbf{C}_n(A;\lambda,a;x)y^n]=(1+y)^n\mathbf{C}_n\left(A;\lambda,a;x\left(1+\frac{1}{y}\right)\right).
$$

Thus, we have obtained the generating matrix function which has the expansion:

$$
(1+y)^n\mathbf{C}_n\left(A;\lambda,a;x\left(1+\frac{1}{y}\right)\right)=\sum_{k=0}^{n}\frac{n!}{(n-k)!k!}(\log(a))^k y^{n-k}\mathbf{C}_{n-k}(A;\lambda,a;x). \tag{3.19}
$$

If we divide by y^n

$$
\left(1+\frac{1}{y}\right)^n\mathbf{C}_n\left(A;\lambda,a;x\left(1+\frac{1}{y}\right)\right)=\sum_{k=0}^{n}\frac{n!}{(n-k)!k!}(\log(a))^k y^{-k}\mathbf{C}_{n-k}(A;\lambda,a;x).
$$

Replacing y^{-1} by t, we get the generating matrix relation

$$(1+t)^{n}\mathbf{C}_{n}\Big(A;\lambda,a;x(1+t)\Big)=\sum_{k=0}^{n}\frac{n!}{(n-k)!k!}(\log(a))^{k}t^{k}\mathbf{C}_{n-k}(A;\lambda,a;x). \quad (3.20)$$

Case 2. From (2.17) for an arbitrary matrix function, we know that

$$e^{c\mathbb{C}}f(x,y,A)=e^{-\lambda cy\log(a)}\Big(1+\frac{cy}{x}\Big)^{A}f\Big(x+cy,y,A\Big).$$

Therefore, for $c=1$, we have

$$\begin{aligned}e^{\mathbb{C}}[\mathbf{C}_{n}(A;\lambda,a;x)y^{n}]&=e^{-\lambda y\log(a)}\Big(1+\frac{y}{x}\Big)^{A}\mathbf{C}_{n}\Big(A;\lambda,a;x+y,y\Big)\\&=y^{n}e^{-\lambda y\log(a)}\Big(1+\frac{y}{x}\Big)^{A}\mathbf{C}_{n}\Big(A;\lambda,a;x+y\Big).\end{aligned}$$

If we equate the two expansions of this matrix function, we get

$$y^{n}e^{-\lambda y\log(a)}\Big(1+\frac{y}{x}\Big)^{A}\mathbf{C}_{n}\Big(A;\lambda,a;x+y\Big)=\sum_{k=0}^{\infty}\frac{(-\lambda)^{k}}{k!}y^{n+k}\mathbf{C}_{n+k}(A;\lambda,a;x). \quad (3.21)$$

If we divide both members of this equation by y^{n} and simplify, we obtain the generating matrix relation:

$$e^{-\lambda y\log(a)}\Big(1+\frac{y}{x}\Big)^{A}\mathbf{C}_{n}\Big(A;\lambda,a;x+y\Big)=\sum_{k=0}^{\infty}\frac{(-\lambda)^{k}}{k!}y^{k}\mathbf{C}_{n+k}(A;\lambda,a;x). \quad (3.22)$$

If $n=0$, this relation becomes the familiar generating matrix functions

$$e^{-\lambda y\log(a)}\Big(1+\frac{y}{x}\Big)^{A}\mathbf{C}_{0}\Big(A;\lambda,a;x+y\Big)=\sum_{k=0}^{\infty}\frac{(-\lambda)^{k}}{k!}y^{k}\mathbf{C}_{k}(A;\lambda,a;x). \quad (3.23)$$

Case 3. For $bc\neq 0$. In order to simplify results we choose $c=1$ and $b=-\frac{1}{\alpha}$. Then for all finite values of α, we have $bc\neq 0$. For any arbitrary matrix function, then, we have

$$\begin{aligned}e^{\mathbb{C}}e^{-\frac{1}{\alpha}\mathbb{B}}f(x,y,A)&=e^{\mathbb{C}}f\Big(x\Big(1-\frac{1}{\alpha y}\Big),y-\frac{1}{\alpha},A\Big)\\&=e^{-\lambda y\log(a)}\Big(1+\frac{y}{x}\Big)^{A}f\Big((x+y)\Big(1-\frac{1}{\alpha y}\Big),y-\frac{1}{\alpha},A\Big).\end{aligned}$$

Hence, we get

$$e^{\mathbb{C}}e^{-\frac{1}{\alpha}\mathbb{B}}[\mathbf{C}_{n}(A;\lambda,a;x)y^{n}]=\Big(y-\frac{1}{\alpha}\Big)^{n}e^{-\lambda y\log(a)}\Big(1+\frac{y}{x}\Big)^{A}\mathbf{C}_{n}\Big(A;\lambda,a;(x+y)\Big(1-\frac{1}{\alpha y}\Big)\Big). \quad (3.24)$$

Separately, we consider the left hand side of (3.24) and we write exponential operators in a series form so that we have the following relation

$$
\begin{aligned}
e^{\mathbb{C}}e^{-\frac{1}{\alpha}\mathbb{B}}[\mathbf{C}_n(A;\lambda,a;x)y^n] &= \sum_{r,k=0}^{\infty}\frac{1}{r!k!}\Big(-\frac{1}{\alpha}\Big)^k\mathbb{C}^r\mathbb{B}^k[\mathbf{C}_n(A;\lambda,a;x)y^n]\\
&= \sum_{r,k=0}^{\infty}\frac{1}{r!k!}\Big(-\frac{1}{\alpha}\Big)^k\Big(-\lambda\Big)^r\mathbb{B}^k[\mathbf{C}_{n+r}(A;\lambda,a;x)y^{n+r}]\\
&= \sum_{k=0}^{\infty}\sum_{r=0}^{n}\frac{1}{r!k!}\Big(-\frac{1}{\alpha}\Big)^k\Big(-\lambda\Big)^r\mathbb{B}^k[\mathbf{C}_n(A;\lambda,a;x)y^n]\\
&= \sum_{k=0}^{\infty}\sum_{r=0}^{n}\frac{1}{r!k!}\Big(-\frac{1}{\alpha}\Big)^k\Big(-\lambda\Big)^r\Big[n\log(a)(n-1)\log(a)\ldots(n-k+1)\log(a)\Big]\\
&\quad\times\mathbb{B}^{k-k}[\mathbf{C}_{n+k}(A;\lambda,a;x)y^{n+k}]\\
&= \sum_{k=0}^{n}\sum_{r=0}^{n}\frac{n!}{(n-k)!r!k!}\Big(-\frac{1}{\alpha}\Big)^k\Big(\log(a)\Big)^k\Big(-\lambda\Big)^r\mathbb{B}^0[\mathbf{C}_{n+k}(A;\lambda,a;x)y^{n+k}]\\
&= \sum_{k=0}^{n}\sum_{r=0}^{n}\frac{n!}{(n-k)!r!k!}\Big(-\frac{1}{\alpha}\Big)^k\Big(\log(a)\Big)^k\Big(-\lambda\Big)^r\mathbf{C}_{n+k}(A;\lambda,a;x)y^{n+k}\\
&= \sum_{k=0}^{n}\sum_{r=0}^{n}\frac{(-1)^k}{r!k!}(-n)_k\Big(-\frac{1}{\alpha}\Big)^k\Big(\log(a)\Big)^k\Big(-\lambda\Big)^r\mathbf{C}_{n+k}(A;\lambda,a;x)y^{n+k}.
\end{aligned}
\tag{3.25}
$$

Using (3.24) and (3.25), we get

$$
\begin{aligned}
&\Big(y-\frac{1}{\alpha}\Big)^n e^{-\lambda y\log(a)}\Big(1+\frac{y}{x}\Big)^A\mathbf{C}_n\Big(A;\lambda,a;(x+y)\Big(1-\frac{1}{\alpha y}\Big)\Big)\\
&\quad= \sum_{k=0}^{n}\sum_{r=0}^{n}\frac{(-1)^k}{r!k!}(-n)_k\Big(-\frac{1}{\alpha}\Big)^k\Big(\log(a)\Big)^k\Big(-\lambda\Big)^r\mathbf{C}_{n+k}(A;\lambda,a;x)y^{n+k}.
\end{aligned}
\tag{3.26}
$$

Finally, we replace y by $-t$ in above equation and simplifying, we obtained the generating matrix function between CMPs, which we believe to be new generating function

$$
\begin{aligned}
&\Big(t+\frac{1}{\alpha}\Big)^n e^{\lambda t\log(a)}\Big(1-\frac{t}{x}\Big)^A\mathbf{C}_n\Big(A;\lambda,a;(x-t)\Big(1+\frac{1}{\alpha t}\Big)\Big)\\
&\quad= \sum_{k=0}^{n}\sum_{r=0}^{n}\frac{(-1)^{r+k}(-n)_k\lambda^r\big(\log(a)\big)^k}{r!k!\alpha^k}\mathbf{C}_{n+k}(A;\lambda,a;x)t^{n+k}.
\end{aligned}
\tag{3.27}
$$

4 Conclusion

A novel approach has been adopted in this work for developing some important properties of CMPs viz recurrence matrix relations, differential recurrence matrix relations and differential matrix equation. Certain generating matrix relations established in this discussion prove the usefulness of operators techniques in obtaining the results for the newly considered CMPs from the corresponding ones holding the unknown and known Charlier polynomials. Other important properties of CMPs will be obtained for further research work. Since the CMPs are significant from the view point of applications. The Lie algebra method developed in this work can be used in the investigation of some other CMPs which play a vital role in Mathematical Physics.

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