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ORIGINAL RESEARCH

Certain new formulas for bibasic Humbert hypergeometric functions Ψ_1 and Ψ_2

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Abstract The main aim of the present work is to give some interesting the q-analogues of various q-recurrence relations, q-recursion formulas, q-partial derivative relations, q-integral representations, transformation and summation formulas for bibasic Humbert hypergeometric functions Ψ_1 and Ψ_2 on two independent bases q and p of two variables, believed to be new, by using the conception of q-calculus. Finally, some interesting special cases and straightforward identities connected with bibasic Humbert hypergeometric series of the types Ψ_1 and Ψ_2 are established when the two independent bases q and p are equal.

Keywords Bibasic series · Bibasic Humbert functions · q-calculus · Summation formulas · Transformation formulas

Mathematics Subject Classification 05A30 · 33D65 · 33D70 · 33D50

1 Introduction

Quantum calculus is the modern name for the investigation of calculus without limits. The quantum calculus or q-calculus began with Jackson in the early twentieth century [16–18], but this kind of calculus had already been worked out by Euler and Jacobi. q-calculus appeared as a connection between mathematics and physics. It has a lot of their applications in many fields of mathematics areas such as number theory, an engineering, combinatorics, orthogonal polynomials, basic hypergeometric functions, quantum theory, physics, mechanics, the theory of relativity and other sciences, see for example [6, 19, 20, 30]. Recently, there have been many studies on q-calculus. Andrews [2], Bytev and Zhang [4], Verma [38], Verma and Sahai [39], Verma and Sarasvati [40] and Yadav et al. [41] discussed recursion formulas and transformations for q-hypergeometric series. Sahai and Verma [22], Sears [23], Srivastava [32, 35] and Upadhyay [37] derived transformations and summation formulas for bilateral basic hypergeometric series. In particular, Jackson [16, 17] was the first to study basic Appell series. Agarwal has developed some properties of basic Appell series [1], and Slater [31] applies contour integral techniques to such series. In a recent paper the authors [7–10, 33, 34, 36] have developed very general transformations involving bibasic q-Appell hypergeometric series on two unconnected bases (q and p, $0 < |q| < 1$, $0 < |p| < 1$, $q, p \in \mathbb{C}$). Such generalized basic Appell hypergeometric series were called bibasic. Earlier, the author in [24–27] have developed and studied some relations of the (p, q)-Bessel, (p, q)-Humbert functions and basic Horn functions H_3, H_4, H_6 and H_7. Motivated by in the previous work [28, 29], many terminating summation and transformation formulas for bibasic q-Humbert hypergeometric series are derived and proved by using

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contiguous relations which extend most of the results due to Shehata and have developed their transformation theory. This paper is concluded by obtaining some q-recurrence relations, q-derivatives formulas, q-partial derivative relations, q-derivatives with respect to the parameters, q-integral representations, transformations and summation formulas for these bibasic Humbert hypergeometric functions Ψ_1 and Ψ_2 with different bases p and q. We think these results are not found in the literature to discuss further consequences of our extensions as special cases.

1.1 Notations and preliminaries

First, we start by remembering some elementary definitions and notations of q-analogue with q-derivative in the q-theory. Throughout this study, unless otherwise is stated, the bases q and p will be assumed to be such that $0 < |q| < 1$, $0 < |p| < 1$, and $q, p \in \mathbb{C}$, for definiteness. We use $\mathbb{C}$ to denote the set of complex numbers and $\mathbb{N}$ to denote the set of positive integers.

For $q \in \mathbb{C}$ and $0 < |q| < 1$, the q-shifted factorials $(q^a; q)_k$ are defined as

$$\begin{aligned}(q^a; q)_k &= \begin{cases} \prod_{r=0}^{k-1}(1 - q^{a+r}), & k \geq 1;\\ 1, & k = 0.\end{cases}\\ &= \begin{cases}(1-q^a)(1-q^{a+1})\dots(1-q^{a+k-1}), & k \in \mathbb{N}, a \in \mathbb{C} \setminus \{0, -1, -2, \dots, 1-k\};\\ 1, & k = 0, a \in \mathbb{C}.\end{cases}\end{aligned} \tag{1.1}$$

Definition 1.1 The q-integer number is defined as

$$_q = \frac{1 - q^{\chi}}{1 - q}, \chi \in \mathbb{N}_0. \tag{1.2}$$

Definition 1.2 For $a, b, c \in \mathbb{C}, c \neq 0, -1, -2, ...$ and $0 < |q| < 1, q \in \mathbb{C}$, the basic hypergeometric series with base q is defined as (see [3,12–15])

$${}_2\phi_1(q^a, q^b; q^c; q, x) = \sum_{k=0}^{\infty} \frac{(q^a; q)_k (q^b; q)_k}{(q^c; q)_k (q; q)_k} x^k, \tag{1.3}$$

for $|x| < 1$ and by analytic continuation for other $x \in \mathbb{C}$.

Definition 1.3 Let f be a function defined on a subset of the complex or real plane. The q-difference operator $\mathbb{D}_{x,q}$ is defined [18] as follows

$$\mathbb{D}_{x,q} f(x) = \frac{f(x) - f(qx)}{(1-q)x}, x \neq 0. \tag{1.4}$$

Definition 1.4 For $0 < |p| < 1, 0 < |q| < 1, p, q \in \mathbb{C}$, we define the bibasic Humbert functions Ψ_1 and Ψ_2 on two independent bases p and q as follows

$$\begin{aligned}\Psi_1(q^a, p^b; p^c, q^d; q, p, x, y) &= \sum_{\ell,k=0}^{\infty} \frac{(q^a; q)_{k+\ell}(p^b; p)_\ell}{(p^c; p)_\ell (q^d; q)_k (q; q)_k (p; p)_\ell} x^k y^\ell,\\ &\quad (p^c, q^d \neq 1, q^{-1}, q^{-2}, \dots, |x|, |y| < 1)\end{aligned} \tag{1.5}$$

and

$$\begin{aligned}\Psi_2(q^a; p^b, q^c; q, p, x, y) &= \sum_{\ell,k=0}^{\infty} \frac{(q^a; q)_{k+\ell}}{(p^b; p)_\ell (q^c; q)_k (q; q)_k (p; p)_\ell} x^k y^\ell,\\ &\quad (q^c, p^b \neq 1, q^{-1}, q^{-2}, \dots, |x|, |y| < 1).\end{aligned} \tag{1.6}$$

2 Main results

In this section, we derive the q-analogues and extensions of bibasic Humbert functions Ψ_1 and Ψ_2 on two independent bases q and p with their several interesting properties on the same order.

Theorem 2.1 *The functions* Ψ_1 *and* Ψ_2 *satisfy the following recurrence relations*

$$\begin{aligned}&\Psi_1(q^{a+1}, p^b; p^c, q^d; q, p, x, y) = \Psi_1(q^a, p^b; p^c, q^d; q, p, x, y) + \frac{q^a x}{1-q^d}\\&\times \Psi_1(q^{a+1}, p^b; p^c, q^{d+1}; q, p, x, y) + \frac{q^a}{1-q^a}\Psi_1(q^a, p^b; p^c, q^d; q, p, qx, y)\\&- \frac{q^a}{1-q^a}\Psi_1(q^a, p^b; p^c, q^d; q, p, qx, qy), q^a, q^d \neq 1,\end{aligned} \tag{2.1}$$

$$\begin{aligned}&\Psi_1(q^{a+1}, p^b; p^c, q^d; q, p, x, y) = \frac{1}{1-q^a}\Psi_1(q^a, p^b; p^c, q^d; q, p, x, y) - \frac{q^a}{1-q^a}\\&\times \Psi_1(q^a, p^b; p^c, q^d; q, p, x, qy) + \frac{q^a x}{1-q^d}\Psi_1(q^{a+1}, p^b; p^c, q^{d+1}; q, p, x, qy), q^a, q^d \neq 1,\\&\Psi_1(q^a, p^b; p^c, q^{d-1}; q, p, x, y) = \Psi_1(q^a, p^b; p^c, q^d; q, p, x, y)\\&+ \frac{q^{d-1}(1-q^a)x}{(1-q^{d-1})(1-q^d)}\Psi_1(q^{a+1}, p^b; p^c, q^{d+1}; q, p, x, y), q^d, q^{d-1} \neq 1\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}&\Psi_2(q^{a+1}; p^b, q^c; q, p, x, y) = \Psi_2(q^a; p^b, q^c; q, p, x, y) + \frac{q^a x}{1-q^c}\Psi_2(q^{a+1}; p^b, q^{c+1}; q, p, x, y)\\&+ \frac{q^a}{1-q^a}\Psi_2(q^a; p^b, q^c; q, p, qx, y) - \frac{q^a}{1-q^a}\Psi_2(q^a; p^b, q^c; q, p, qx, qy), q^a, q^c \neq 1,\\&\Psi_2(q^{a+1}; p^b, q^c; q, p, x, y) = \frac{1}{1-q^a}\Psi_2(q^a; p^b, q^c; q, p, x, y) - \frac{q^a}{1-q^a}\\&\times \Psi_2(q^a; p^b, q^c; q, p, x, qy) + \frac{q^a x}{1-q^c}\Psi_2(q^{a+1}; p^b, q^{c+1}; q, p, x, qy), q^a, q^c \neq 1,\\&\Psi_2(q^a; p^b, q^{c-1}; q, p, x, y) = \Psi_2(q^a; p^b, q^c; q, p, x, y)\\&+ \frac{q^{c-1}(1-q^a)x}{(1-q^{c-1})(1-q^c)}\Psi_2(q^{a+1}; p^b, q^{c+1}; q, p, x, y), q^c, q^{c-1} \neq 1.\end{aligned} \tag{2.3}$$

Proof To prove the identity (2.1). Using the relations

$$\begin{aligned}(q^{a+1}; q)_{k+\ell} =& \Big[1 + q^a \frac{1-q^{k+\ell}}{1-q^a}\Big](q^a; q)_{k+\ell},\\(q^a; q)_{k+\ell+1} =& (1-q^a)(q^{a+1}; q)_{k+\ell},\end{aligned}$$

and

$$\begin{aligned}1 - q^{k+\ell} =& 1 - q^k + q^k - q^{k+\ell},\\=& 1 - q^\ell + q^\ell(1-q^k),\end{aligned}$$

we have

$$
\begin{aligned}
&\Psi_1(q^{a+1}, p^b; p^c, q^d; q, p, x, y) - \Psi_1(q^a, p^b; p^c, q^d; q, p, x, y)\\
&\quad = \frac{q^a}{1-q^a} \sum_{\ell=0,k=1}^{\infty} \frac{(q^a;q)_{k+\ell}(p^b;p)_\ell}{(p^c;p)_\ell(q^d;q)_k(q;q)_{k-1}(p;p)_\ell} x^k y^\ell\\
&\qquad + \frac{q^a}{1-q^a} \sum_{\ell,k=0}^{\infty} \frac{q^k(q^a;q)_{k+\ell}(p^b;p)_\ell}{(p^c;p)_\ell(q^d;q)_k(q;q)_k(p;p)_\ell} x^k y^\ell\\
&\qquad - \frac{q^a}{1-q^a} \sum_{\ell,k=0}^{\infty} \frac{q^{k+\ell}(q^a;q)_{k+\ell}(p^b;p)_\ell}{(p^c;p)_\ell(q^d;q)_k(q;q)_k(p;p)_\ell} x^k y^\ell\\
&\quad = \frac{q^a}{1-q^a} \sum_{\ell,k=0}^{\infty} \frac{(q^a;q)_{k+\ell+1}(p^b;p)_\ell}{(p^c;p)_\ell(q^d;q)_{k+1}(q;q)_k(p;p)_\ell} x^{k+1} y^\ell\\
&\qquad + \frac{q^a}{1-q^a} \sum_{\ell,k=0}^{\infty} \frac{(q^a;q)_{k+\ell}(p^b;p)_\ell}{(p^c;p)_\ell(q^d;q)_k(q;q)_k(p;p)_\ell} (qx)^k y^\ell\\
&\qquad - \frac{q^a}{1-q^a} \sum_{\ell,k=0}^{\infty} \frac{(q^a;q)_{k+\ell}(p^b;p)_\ell}{(p^c;p)_\ell(q^d;q)_k(q;q)_k(p;p)_\ell} (qx)^k (qy)^\ell\\
&\quad = \frac{q^a x}{1-q^d} \Psi_1(q^{a+1}, p^b;; p^c, q^{d+1}; q, p, x, y) + \frac{q^a}{1-q^a} \Psi_1(q^a, p^b; p^c, q^d; q, p, qx, y)\\
&\qquad - \frac{q^a}{1-q^a} \Psi_1(q^a, p^b; p^c, q^d; q, p, qx, qy).
\end{aligned}
$$

The proofs of the relations (2.2) and (2.3) follow in the same way. □

Theorem 2.2 *The following relations for Ψ_1 and Ψ_2 are true*

$$
\begin{aligned}
&\Psi_1(q^a, p^{b+1}; p^c, q^d; q, p, x, y) = \Psi_1(q^a, p^b; p^c, q^d; q, p, x, y)\\
&\quad + \frac{p^b(1-q^a)y}{1-p^c} \Psi_1(q^{a+1}, p^{b+1}; p^{c+1}, q^d; q, p, x, y), \; p^c \neq 1,
\end{aligned} \tag{2.4}
$$

$$
\begin{aligned}
&\Psi_1(q^a, p^b; p^{c-1}, q^d; q, p, x, y) = \Psi_1(q^a, p^b; p^c, q^d; q, p, x, y)\\
&\quad + \frac{p^{c-1}(1-q^a)(1-p^b)y}{(1-p^{c-1})(1-p^c)} \Psi_1(q^{a+1}, p^{b+1}; p^{c+1}, q^d; q, p, x, y), \; p^c, p^{c-1} \neq 1,\\
&\Psi_1(q^a, p^b; p^c, q^d; q, p, x, y) = (1-p^b)\Psi_1(q^a, p^{b+1}; p^c, q^d; q, p, x, y)\\
&\quad + p^b \Psi_1(q^a, p^b; p^c, q^d; q, p, x, py)
\end{aligned} \tag{2.5}
$$

and

$$
\begin{aligned}
&\Psi_2(q^a; p^{b-1}, q^c; q, p, x, y) = \Psi_2(q^a; p^b, q^c; q, p, x, y)\\
&\quad + \frac{p^{b-1}(1-q^a)y}{(1-p^{b-1})(1-p^b)} \Psi_2(q^{a+1}; p^{b+1}, q^c; q, p, x, y), \; p^b, p^{b-1} \neq 1.
\end{aligned} \tag{2.6}
$$

Proof To prove the relation (2.4). Using the relation

$$
(p^b; p)_{\ell+1} = (1-p^b)(p^{b+1}; p)_\ell = (1-p^{b+\ell})(p^b; p)_\ell,
$$

and (1.5), we have

$$\begin{aligned}
&\Psi_1(q^a, p^{b+1}; p^c, q^d; q, p, x, y) - \Psi_1(q^a, p^b; p^c, q^d; q, p, x, y)\\
&= p^b \sum_{\ell,k=0}^{\infty} [\frac{1-p^\ell}{1-p^b}] \frac{(q^a;q)_{k+\ell}(p^b;p)_\ell}{(p^c;p)_\ell(q^d;q)_k(q;q)_k(p;p)_\ell} x^k y^\ell\\
&= \frac{p^b}{1-p^b} \sum_{\ell=1,k=0}^{\infty} \frac{(q^a;q)_{k+\ell}(p^b;p)_\ell}{(p^c;p)_\ell(q^d;q)_k(q;q)_k(p;p)_{\ell-1}} x^k y^\ell\\
&= \frac{p^b(1-q^a)y}{1-p^c} \sum_{\ell,k=0}^{\infty} \frac{(q^{a+1};q)_{k+\ell}(p^{b+1};p)_\ell}{(p^{c+1};p)_\ell(q^d;q)_k(q;q)_k(p;p)_\ell} x^k y^{\ell+1}\\
&= \frac{p^b(1-q^a)y}{1-p^c} \Psi_1(q^{a+1}, p^{b+1}; p^{c+1}, q^d; q, p, x, y),\ p^c \neq 1.
\end{aligned}$$

A similar way to the proof of relation (2.4), we obtain the results (2.5) and (2.6). □

Theorem 2.3 *The relations for Ψ_1 and Ψ_2 hold true*

$$\begin{aligned}
&(1-q^a)\Psi_1\left(q^{a+1}, p^b; p^c, q^d; q, p, x, \frac{y}{q}\right) + q^{a+1-d}\Psi_1(q^a, p^b; p^c, q^d; q, p, x, y)\\
&= \Psi_1\left(q^a, p^b; p^c, q^d; q, p, x, \frac{y}{q}\right) + q^{a+1-d}(1-q^{d-1})\Psi_1(q^a, p^b; q^{d-1}; q, p, x, y),
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
(1-p^b)\Psi_1(q^a, p^{b+1}; p^c, q^d; q, p, x, y) &= p^{b+1-c}(1-p^{c-1})\Psi_1(q^a, p^b; p^{c-1}, q^d; q, p, x, y)\\
&+ (1-p^{b+1-c})\Psi_1(q^a, p^b; p^c, q^d; q, p, x, y)
\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}
&(1-q^a)\Psi_2\left(q^{a+1}; p^b, q^c; q, p, x, \frac{y}{q}\right) + q^{a+1-c}\Psi_2(q^a; p^b, q^c; q, p, x, y)\\
&= \Psi_2\left(q^a; p^b, q^c; q, p, x, \frac{y}{q}\right) + q^{a+1-c}(1-q^{c-1})\Psi_2(q^a; p^b, q^{c-1}; q, p, x, y).
\end{aligned} \tag{2.9}$$

Proof Using the relationship

$$\frac{1-q^{d-1}}{(q^{d-1};q)_k} = \frac{1-q^{d+k-1}}{(q^d;q)_k} = \frac{1}{(q^d;q)_{k-1}},$$

we get

$$\begin{aligned}
&q^{a+1-d}(1-q^{d-1})\Psi_1(q^a, p^b; p^c, q^{d-1}; q, p, x, y) = \sum_{\ell,k=0}^{\infty} \frac{(q^{a+1-d}-q^{a+k})(q^a;q)_{k+\ell}(p^b;p)_\ell}{(p^c;p)_\ell(q^d;q)_k(q;q)_k(p;p)_\ell} x^k y^\ell\\
&= \sum_{\ell,k=0}^{\infty} \frac{(1-q^{a+k+\ell})(q^a;q)_{k+\ell}(p^b;p)_\ell}{(p^c;p)_\ell(q^d;q)_k(q;q)_k(p;p)_\ell} x^k \left(\frac{y}{q}\right)^\ell - \sum_{\ell,k=0}^{\infty} \frac{(q^{-\ell}-q^{a+1-d})(q^a;q)_{k+\ell}(p^b;p)_\ell}{(p^c;p)_\ell(q^d;q)_k(q;q)_k(p;p)_\ell} x^k y^\ell\\
&= (1-q^a)\Psi_1\left(q^{a+1}, p^b; p^c, q^d; q, p, x, \frac{y}{q}\right) - \Psi_1\left(q^a, p^b; p^c, q^d; q, p, x, \frac{y}{q}\right)\\
&\quad + q^{a+1-d}\Psi_1(q^a, p^b; p^c, q^d; q, p, x, y).
\end{aligned}$$

A similar argument, we obtain the relations (2.8) and (2.9). □

Theorem 2.4 *For $r, s \in \mathbb{N}$, the bibasic Humbert functions Ψ_1 and Ψ_2 satisfy the q and p-difference equations*

$$\mathbb{D}_{x,q}^r \Psi_1(q^a, p^b; p^c, q^d; q, p, x, y) = \frac{(q^a;q)_r}{(1-q)^r(q^d;q)_r} \Psi_1(q^{a+r}, p^b; p^c, q^{d+r}; q, p, x, y), \tag{2.10}$$

$$\mathbb{D}^{s}_{y,p}\Psi_1(q^a,p^b;p^c,q^d;q,p,x,y)=\frac{(q^a;q)_s(p^b;p)_r}{(1-p)^s(p^c;p)_s}\Psi_1(q^{a+s},p^{b+s};p^{c+s},q^d;q,p,x,y), \tag{2.11}$$

$$\begin{aligned}\mathbb{D}^{r}_{x,p}\mathbb{D}^{s}_{y,p}\Psi_1(q^a,p^b;p^c,q^d;q,p,x,y)=&\frac{(q^a;q)_{r+s}(p^b;p)_r}{(1-q)^r(1-p)^s(p^c;p)_s(q^d;q)_r}\\&\times\Psi_1(q^{a+r+s},p^{b+s};p^{c+s},q^{d+r};q,p,x,y)\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}\mathbb{D}^{r}_{x,q}\Psi_2(q^a;p^b,q^c;q,p,x,y)&=\frac{(q^a;q)_r}{(1-q)^r(q^c;q)_r}\Psi_2(q^{a+r};p^b,q^{c+r};q,p,x,y),\\ \mathbb{D}^{s}_{y,p}\Psi_2(q^a;p^b,q^c;q,p,x,y)&=\frac{(q^a;q)_s}{(1-p)^s(p^b;p)_s}\Psi_2(q^{a+s};p^{b+s},q^c;q,p,x,y),\\ \mathbb{D}^{r}_{x,p}\mathbb{D}^{s}_{y,p}\Psi_2(q^a;p^b,q^c;q,p,x,y)&=\frac{(q^a;q)_{r+s}\Psi_2(q^{a+r+s};p^{b+s},q^{c+r};q,p,x,y)}{(1-q)^r(1-p)^s(p^b;p)_s(q^c;q)_r}.\end{aligned} \tag{2.13}$$

Proof Calculating the q-derivative of both sides of (1.5) with respect to the variable x, we get

$$\begin{aligned}\mathbb{D}_{x,q}\Psi_1(q^a,p^b;p^c,q^d;q,p,x,y)&=\sum_{\ell=0,k=1}^{\infty}\frac{1-q^k}{1-q}\frac{(q^a;q)_{k+\ell}(p^b;p)_\ell}{(p^c;p)_\ell(q^d;q)_k(q;q)_k(p;p)_\ell}x^{k-1}y^\ell\\&=\frac{(1-q^a)}{(1-q)(1-q^d)}\sum_{\ell,k=0}^{\infty}\frac{(q^{a+1};q)_{k+\ell}(p^b;p)_\ell}{(p^c;p)_\ell(q^{d+1};q)_k(q;q)_k(p;p)_\ell}x^ky^\ell\\&=\frac{(1-q^a)}{(1-q)(1-q^d)}\Psi_1(q^{a+1},p^b;p^c,q^{d+1};q,p,x,y).\end{aligned} \tag{2.14}$$

Iterating this q-derivative on Ψ_1 for r-times in the above Eq. (2.14), we obtain (2.10).

Using the p-derivative in (1.5), we get

$$\begin{aligned}\mathbb{D}_{y,p}\Psi_1(q^a,p^b;p^c,q^d;q,p,x,y)&=\sum_{\ell=1,k=0}^{\infty}\frac{1}{1-p}\frac{(q^a;q)_{k+\ell}(p^b;p)_\ell}{(p^c;p)_\ell(q^d;q)_k(p;p)_{\ell-1}(q;q)_k}x^ky^{\ell-1}\\&=\frac{(1-q^a)(1-p^b)}{(1-p)(1-p^c)}\sum_{\ell,k=0}^{\infty}\frac{(q^{a+1};q)_{k+\ell}(p^{b+1};p)_\ell}{(p^{c+1};p)_\ell(q^d;q)_k(q;q)_k(p;p)_\ell}x^ky^\ell\\&=\frac{(1-q^a)(1-p^b)}{(1-p)(1-p^c)}\Psi_1(q^{a+1},p^{b+1};p^{c+1};q,p,x,y).\end{aligned} \tag{2.15}$$

Iterating this p-derivative on Ψ_1 for s-times in the above equation (2.15), we obtain (2.11). Similarly, the q-derivatives given by (1.4) can be proved (2.12)–(2.13). □

Theorem 2.5 *The q-differential relations for Ψ_1 and Ψ_2 hold*

$$\begin{aligned}x\mathbb{D}_{x,q}\Psi_1(q^a,p^b;p^c,q^d;q,p,x,y)=&\frac{(1-q^{d-1})}{(1-q)q^{d-1}}\big[\Psi_1(q^a,p^b;p^c,q^{d-1};q,p,x,y)\\&-\Psi_1(q^a,p^b;p^c,q^d;q,p,x,y)\big],\end{aligned} \tag{2.16}$$

$$\begin{aligned}y\mathbb{D}_{y,p}\Psi_1(q^a,p^b;p^c,q^d;q,p,x,y)=&\frac{1-p^b}{(1-p)p^b}\big[\Psi_1(q^a,p^{b+1};p^c,q^d;q,p,x,y)\\&-\Psi_1(q^a,p^b;p^c,q^d;q,p,x,y)\big],\\ y\mathbb{D}_{y,p}\Psi_1(q^a,p^b;p^c,q^d;q,p,x,y)=&\frac{1-p^{c-1}}{(1-p)p^{c-1}}\big[\Psi_1(q^a,p^b;p^{c-1},q^d;q,p,x,y)\\&-\Psi_1(q^a,p^b;p^c,q^d;q,p,x,y)\big]\end{aligned} \tag{2.17}$$

and

$$\begin{aligned} x\mathbb{D}_{x,q}\Psi_2(q^a;p^b,q^c;q,p,x,y) =& \frac{(1-q^{c-1})}{(1-q)q^{c-1}}\Big[\Psi_2(q^a;p^b,q^{c-1};q,p,x,y) \\ &- \Psi_2(q^a;p^b,q^c;q,p,x,y)\Big], \\ y\mathbb{D}_{y,p}\Psi_2(q^a;p^b,q^c;q,p,x,y) =& \frac{1-p^{b-1}}{(1-p)p^{b-1}}\Big[\Psi_2(q^a;p^{b-1},q^c;q,p,x,y) \\ &- \Psi_2(q^a;p^b,q^c;q,p,x,y)\Big]. \end{aligned} \tag{2.18}$$

Proof Multiplying (2.14) by x and substituting the value of $\Psi_1(q^{a+1},p^b;p^c,q^{d+1};q,p,x,y)$ from (2.2) into (2.13), we get (2.16).

The proofs (2.17)–(2.18) are similar to the proof of the corresponding identity (2.16) for the bibasic Humbert functions Ψ_1 and Ψ_2 are omitted most steps and give only outlines wherever necessary. □

Theorem 2.6 *The following relations for Ψ_1 and Ψ_2 hold:*

$$\begin{aligned} &(1-q^{d-1})\Psi_1(q^a,p^b;p^c,q^{d-1};q,p,x,y) = (1-q)x\mathbb{D}_{x,q}\Psi_1(q^a,p^b;p^c,q^d;q,p,x,y) \\ &\quad +(1-q^{d-1})\Psi_1(q^a,p^b;p^c,q^d;q,p,qx,y), \end{aligned} \tag{2.19}$$

$$\begin{aligned} &(1-p^{c-1})\Psi_1(q^a,p^b;p^{c-1},q^d;q,p,x,y) = (1-p)y\mathbb{D}_{y,p}\Psi_1(q^a,p^b;p^c,q^d;q,p,x,y) \\ &\quad +(1-p^{c-1})\Psi_1(q^a,p^b;p^c,q^d;q,p,x,py), \\ &(1-p^b)\Psi_1(q^a,p^{b+1};p^c,q^d;q,p,x,y) = (1-p)y\mathbb{D}_{y,p}\Psi_1(q^a,p^b;p^c,q^d;q,p,x,y) \\ &\quad +(1-p^b)\Psi_1(q^a,p^b;p^c,q^d;q,p,x,py), \end{aligned} \tag{2.20}$$

$$\begin{aligned} &(1-q^a)\Psi_1(q^{a+1},p^b;p^c,q^d;q,p,x,xy) = (1-q^a)\Psi_1(q^a,p^b;p^c,q^d;q,p,x,xy) \\ &\quad +(1-q)q^a x\mathbb{D}_{x,q}\Psi_1(q^a,p^b;p^c,q^d;q,p,x,xy), \\ &(1-q^a)\Psi_1(q^{a+1},p^b;p^c,q^d;q,p,x,xy) = (1-q)x\mathbb{D}_{x,q}\Psi_1(q^a,p^b;p^c,q^d;q,p,x,xy) \\ &\quad +(1-q^a)\Psi_1(q^a,p^b;p^c,q^d;q,p,qx,qxy), \end{aligned} \tag{2.21}$$

$$\begin{aligned} &(1-q^a)\Psi_1(q^{a+1},p^b;p^c,q^d;q,p,xy,y) = (1-q^a)\Psi_1(q^a,p^b;p^c,q^d;q,p,xy,y) \\ &+(1-q)q^a y\mathbb{D}_{y,q}\Psi_1(q^a,p^b;p^c,q^d;q,p,xy,y), \\ &(1-q^a)\Psi_1(q^{a+1},p^b;p^c,q^d;q,p,xy,y) = (1-q)y\mathbb{D}_{y,q}\Psi_1(q^a,p^b;p^c,q^d;q,p,xy,y) \\ &\quad +(1-q^a)\Psi_1(q^a,p^b;p^c,q^d;q,p,qxy,qy), \end{aligned} \tag{2.22}$$

$$\begin{aligned} &(1-q^{c-1})\Psi_2(q^a;p^c,q^{c-1};q,p,x,y) = (1-q)x\mathbb{D}_{x,q}\Psi_2(q^a;p^b,q^c;q,p,x,y) \\ &\quad +(1-q^{c-1})\Psi_2(q^a;p^b,q^c;q,p,qx,y), \\ &(1-p^{b-1})\Psi_2(q^a;p^{b-1},q^c;q,p,x,y) = (1-p)y\mathbb{D}_{y,p}\Psi_2(q^a;p^b,q^c;q,p,x,y) \\ &\quad +(1-p^{b-1})\Psi_2(q^a;p^b,q^c;q,p,x,py), \end{aligned} \tag{2.23}$$

$$\begin{aligned} &(1-q^a)\Psi_2(q^{a+1};p^b,q^c;q,p,x,xy) = (1-q^a)\Psi_2(q^a;p^b,q^c;q,p,x,xy) \\ &\quad +(1-q)q^a x\mathbb{D}_{x,q}\Psi_2(q^a;p^b,q^c;q,p,x,xy), \\ &(1-q^a)\Psi_2(q^{a+1};p^b,q^c;q,p,x,xy) = (1-q)x\mathbb{D}_{x,q}\Psi_2(q^a;p^b,q^c;q,p,x,xy) \\ &\quad +(1-q^a)\Psi_2(q^a;p^b,q^c;q,p,qx,qxy) \end{aligned} \tag{2.24}$$

and

$$\begin{aligned} &(1-q^a)\Psi_2(q^{a+1};p^b,q^c;q,p,xy,y) = (1-q^a)\Psi_2(q^a;p^b,q^c;q,p,xy,y) \\ &\quad + (1-q)q^a y\mathbb{D}_{y,q}\Psi_2(q^a;p^b,q^c;q,p,xy,y), \\ &(1-q^a)\Psi_2(q^{a+1};p^b,q^c;q,p,xy,y) = (1-q)y\mathbb{D}_{y,q}\Psi_2(q^a;p^b,q^c;q,p,xy,y) \\ &\quad + (1-q^a)\Psi_2(q^a;p^b,q^c;q,p,qxy,qy). \end{aligned} \tag{2.25}$$

Proof Using (1.5) and (2.14), we obtain

$$
\begin{aligned}
(1-q^{d-1})\Psi_1(q^a,p^b;p^c,q^{d-1};q,p,x,y) &= \sum_{\ell,k=0}^{\infty}\frac{(q^a;q)_{k+\ell}(p^b;p)_\ell}{(p^c;p)_\ell(q^d;q)_{k-1}(q;q)_k(p;p)_\ell}x^k y^\ell\\
&= \sum_{\ell,k=0}^{\infty}\frac{(1-q^{d+k-1})(q^a;q)_{k+\ell}(p^b;p)_\ell}{(p^c;p)_\ell(q^d;q)_k(q;q)_k(p;p)_\ell}x^k y^\ell\\
&= (1-q)x\mathbb{D}_{x,q}\Psi_1(q^a,p^b;p^c,q^d;q,p,x,y)+(1-q^{d-1})\Psi_1(q^a,p^b;p^c,q^d;q,p,qx,y).
\end{aligned}
$$

The proofs results (2.20)–(2.25) for Ψ_1 and Ψ_2 follows similarly from the identity (2.19), and is omitted most steps and give only outlines wherever necessary. □

Theorem 2.7 *The q-derivatives of bibasic Humbert functions Ψ_1 and Ψ_2 with respect to their parameters satisfies the relations*

$$
\begin{aligned}
\mathbb{D}_{a,q}\Psi_1(q^a,p^b;p^c,q^d;q,p,x,y) &= -\frac{1}{1-q^a}\Big[x\mathbb{D}_{x,q}\Psi_1+\frac{1-p}{1-q}y\mathbb{D}_{y,p}\\
&\quad+\frac{1}{1-q}\Psi_1(qx,py)-\frac{1}{1-q}\Psi_1(qx,qy)\Big],\ q^a\neq 1, \qquad (2.26)\\
\mathbb{D}_{a,q}\Psi_1(q^a,p^b;p^c,q^d;q,p,x,y) &= -\frac{1}{1-q^a}\Big[\frac{1-p}{1-q}y\mathbb{D}_{y,p}\Psi_1+x\mathbb{D}_{x,q}\Psi_1(qy)\\
&\quad+\frac{1}{1-q}\Psi_1(py)-\frac{1}{1-q}\Psi_1(qy)\Big],\ q^a\neq 1,\\
\mathbb{D}_{d,q}\Psi_1(q^a,p^b;p^c,q^d;q,p,x,y) &= \frac{1}{1-q^d}x\mathbb{D}_{x,q}\Psi_1(q^{d+1}),\ q^d\neq 1, \qquad (2.27)\\
\mathbb{D}_{b,p}\Psi_1(q^a,p^b;p^c,q^d;q,p,x,y) &= -\frac{1}{1-p^b}y\mathbb{D}_{y,p}\Psi_1,\ p^b\neq 1,\\
\mathbb{D}_{c,p}\Psi_1(q^a,p^b;p^c,q^d;q,p,x,y) &= \frac{1}{1-p^c}y\mathbb{D}_{y,p}\Psi_1(p^{c+1}),\ p^c\neq 1
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{D}_{a,q}\Psi_2(q^a;p^b,q^c;q,p,x,y) &= -\frac{1}{1-q^a}\Big[x\mathbb{D}_{x,q}\Psi_2+\frac{1-p}{1-q}y\mathbb{D}_{y,p}\Psi_2(qx)\\
&\quad+\frac{1}{1-q}\Psi_2(qx,py)-\frac{1}{1-q}\Psi_2(qx,qy)\Big],\ q^a\neq 1,\\
\mathbb{D}_{a,q}\Psi_2(q^a;p^b,q^c;q,p,x,y) &= -\frac{1}{1-q^a}\Big[\frac{1-p}{1-q}y\mathbb{D}_{y,p}\Psi_2+x\mathbb{D}_{x,q}\Psi_2(qy)\\
&\quad+\frac{1}{1-q}\Psi_2(py)-\frac{1}{1-q}\Psi_2(qy)\Big],\ q^a\neq 1, \qquad (2.28)\\
\mathbb{D}_{c,q}\Psi_2(q^a;p^b,q^c;q,p,x,y) &= \frac{1}{1-q^c}x\mathbb{D}_{x,q}\Psi_2(q^{c+1}),\ q^c\neq 1,\\
\mathbb{D}_{b,p}\Psi_2(q^a;p^b,q^c;q,p,x,y) &= \frac{1}{1-p^b}y\mathbb{D}_{y,p}\Psi_2(p^{b+1}),\ p^b\neq 1.
\end{aligned}
$$

Proof Calculating the q-derivative of both sides of (1.5) with respect to the variable a, we get

$$\begin{aligned}
\mathbb{D}_{a,q}\Psi_1(q^a, p^b; p^c, q^d; q, p, x, y) &= \sum_{\ell,k=0}^{\infty} \frac{(q^a;q)_{k+\ell} - (q^{a+1};q)_{k+\ell}}{(1-q)q^a} \frac{(p^b;p)_\ell}{(p^c;p)_\ell (q^d;q)_k (q;q)_k (p;p)_\ell} x^k y^\ell \\
&= -\frac{1}{1-q^a} \sum_{\ell,k=0}^{\infty} \frac{1-q^{k+\ell}}{1-q} \frac{(q^a;q)_{k+\ell}(p^b;p)_\ell}{(p^c;p)_\ell (q^d;q)_k (q;q)_k (p;p)_\ell} x^k y^\ell \\
&= -\frac{1}{1-q^a} \sum_{\ell,k=0}^{\infty} \left[\frac{1-q^k}{1-q} + \frac{q^k(1-p)}{1-q}\frac{1-p^\ell}{1-p} + q^k \frac{p^\ell - q^\ell}{1-q}\right] \frac{(q^a;q)_{k+\ell}(p^b;p)_\ell}{(p^c;p)_\ell (q^d;q)_k (q;q)_k (p;p)_\ell} x^k y^\ell \\
&= -\frac{1}{1-q^a}\left[x\mathbb{D}_{x,q}\Psi_1 + \frac{1-p}{1-q} y\mathbb{D}_{y,p}\Psi_1(qx) + \frac{1}{1-q}\Psi_1(qx, py) - \frac{1}{1-q}\Psi_1(qx, qy)\right].
\end{aligned}$$

The proofs (2.27)–(2.28) for Ψ_1 and Ψ_2 are similar to the proof of result (2.26) and have been omitted. □

Theorem 2.8 *For $r \in \mathbb{N}$, the differentiation formulas for Ψ_1 and Ψ_2 hold true*

$$\mathbb{D}^r_{y,p}\left[y^{b+r-1}\Psi_1(q^a, p^b; p^c, q^d; q, p, x, y)\right] = \frac{(p^b;p)_r}{(1-p)^r} y^{b-1}\Psi_1(q^a, p^{b+r}; p^c, q^d; q, p, x, y), \tag{2.29}$$

$$\begin{aligned}
\mathbb{D}^r_{x,q}\left[x^{a+r-1}\Psi_1(q^a, p^b; p^c, q^d; q, p, x, xy)\right] &= \frac{(q^a;q)_r}{(1-q)^r} x^{a-1}\Psi_1(q^{a+r}, p^b; p^c, q^d; q, p, x, xy), \\
\mathbb{D}^r_{y,q}\left[y^{a+r-1}\Psi_1(q^a, p^b; p^c, q^d; q, p, xy, y)\right] &= \frac{(q^a;q)_r}{(1-q)^r} y^{a-1}\Psi_1(q^{a+r}, p^b; p^c, q^d; q, p, xy, y)
\end{aligned} \tag{2.30}$$

and

$$\begin{aligned}
\mathbb{D}^r_{x,q}\left[x^{a+r-1}\Psi_2(q^a; p^b, q^c; q, p, x, xy)\right] &= \frac{(q^a;q)_r}{(1-q)^r} x^{a-1}\Psi_2(q^{a+r}; p^b, q^c; q, p, x, xy), \\
\mathbb{D}^r_{y,q}\left[y^{a+r-1}\Psi_2(q^a; p^b, q^c; q, p, xy, y)\right] &= \frac{(q^a;q)_r}{(1-q)^r} y^{a-1}\Psi_2(q^{a+r}; p^b, q^c; q, p, xy, y).
\end{aligned} \tag{2.31}$$

Proof Using

$$\mathbb{D}^r_{y,p}\left[y^{b+\ell+r-1}\right] = \frac{(p^{b+\ell};p)_r}{(1-p)^r} y^{b+\ell-1},$$

and

$$(p^b;p)_{\ell+r} = (p^b;p)_\ell (p^{b+\ell};p)_r = (p^b;p)_r (p^{b+r};p)_\ell,$$

we get

$$\begin{aligned}
&\mathbb{D}^r_{y,p}\left[y^{b+r-1}\Psi_1(q^a, p^b; p^c, q^d; q, p, x, y)\right] \\
&= \frac{1}{(1-p)^r} y^{b-1} \sum_{\ell,k=0}^{\infty} \frac{(q^a;q)_{k+\ell}(p^b;p)_\ell (p^{b+\ell};p)_r}{(p^c;p)_\ell (q^d;q)_k (q;q)_k (p;p)_\ell} x^k y^\ell \\
&= \frac{(p^b;p)_r}{(1-p)^r} y^{b-1}\Psi_1(q^a, p^{b+r}; p^c, q^d; q, p, x, y).
\end{aligned}$$

The proof Eqs. (2.30)–(2.31) are on the same lines as of Eq. (2.29) . □

Theorem 2.9 *The summation formulas for* Ψ_1 *and* Ψ_2 *hold true*

$$\Psi_1(q^a, p^b; p^c, q^d; q, p, x, y) = \sum_{\ell=0}^{\infty} \frac{(q^a; q)_\ell (p^b; p)_\ell}{(p^c; p)_\ell (p; p)_\ell} y^\ell \, {}_2\phi_1(q^{a+\ell}, 0; q^d; q, x) \tag{2.32}$$

and

$$\Psi_2(q^a; p^b, q^c; q, p, x, y) = \sum_{\ell=0}^{\infty} \frac{(q^a; q)_\ell}{(p^b; p)_\ell (p; p)_\ell} y^\ell \, {}_2\phi_1(q^{a+\ell}, 0; q^c; q, x). \tag{2.33}$$

Proof Using the following relationship

$$(q^a; q)_{\ell+k} = (q^a; q)_\ell (q^{a+\ell}; q)_k = (q^a; q)_k (q^{a+k}; q)_\ell,$$

we get

$$\Psi_1(q^a, p^b; p^c, q^d; q, p, x, y) = \sum_{\ell,k=0}^{\infty} \frac{(q^a; q)_\ell (q^{a+\ell}; q)_k (p^b; p)_\ell}{(p^c; p)_\ell (q^d; q)_k (q; q)_k (p; p)_\ell} x^k y^\ell$$

$$= \sum_{\ell=0}^{\infty} \frac{(q^a; q)_\ell (p^b; p)_\ell}{(p^c; p)_\ell (p; p)_\ell} y^\ell \sum_{k=0}^{\infty} \frac{(q^{a+\ell}; q)_k}{(q^d; q)_k (q; q)_k} x^k = \sum_{\ell=0}^{\infty} \frac{(q^a; q)_\ell (p^b; p)_\ell}{(p^c; p)_\ell (p; p)_\ell} y^\ell \, {}_2\phi_1(q^{a+\ell}, 0; q^d; q, x),$$

As it is obtained the Eq. (2.32) can be proven in similarly. □

Theorem 2.10 *The connection relation between bibasic Humbert functions* Ψ_1 *and* Ψ_2 *is shown as follows*

$$\Psi_1(q^a, p^b; p^c, q^d; q, p, x, y) = \frac{(p^b; p)_\infty}{(p^c; p)_\infty} \sum_{s=0}^{\infty} \frac{(p^{c-b}; p)_s p^{bs}}{(p; p)_s} \Psi_2(q^a; 0, q^d; q, p, x, p^s y). \tag{2.34}$$

Proof Using the identities

$$(p^b; p)_\ell = \frac{(p^b; p)_\infty}{(p^{b+\ell}; q)_\infty},$$

$$(p^c; p)_\ell = \frac{(p^c; p)_\infty}{(p^{c+\ell}; q)_\infty},$$

and

$$\frac{(p^{c+\ell}; p)_\infty}{(p^{b+\ell}; p)_\infty} = \sum_{s=0}^{\infty} \frac{(p^{c-b}; p)_s}{(p; p)_s} \left(p^{b+\ell}\right)^s,$$

substitution of this gives

$$\Psi_1(q^a, p^b; p^c, q^d; q, p, x, y) = \frac{(p^b; p)_\infty}{(p^c; p)_\infty} \sum_{\ell,k=0}^{\infty} \frac{(q^a; q)_{k+\ell} (p^{c+\ell}; p)_\infty}{(p^{b+\ell}; p)_\infty (q^d; q)_k (q; q)_k (p; p)_\ell} x^k y^\ell$$

$$= \frac{(p^b; p)_\infty}{(p^c; p)_\infty} \sum_{\ell,k,s=0}^{\infty} \frac{(q^a; q)_{k+\ell} (p^{c-b}; p)_s}{(q^d; q)_k (q; q)_k (p; p)_\ell (p; p)_s} \left(p^{b+\ell}\right)^s x^k y^\ell$$

$$= \frac{(p^b; p)_\infty}{(p^c; p)_\infty} \sum_{s=0}^{\infty} \frac{(p^{c-b}; p)_s p^{bs}}{(p; p)_s} \sum_{\ell,k=0}^{\infty} \frac{(q^a; q)_{k+\ell}}{(q^d; q)_k (q; q)_k (p; p)_\ell} p^{s\ell} x^k y^\ell$$

$$= \frac{(p^b; p)_\infty}{(p^c; p)_\infty} \sum_{s=0}^{\infty} \frac{(p^{c-b}; p)_s p^{bs}}{(p; p)_s} \Psi_2(q^a; 0, q^d; q, p, x, p^s y).$$

□

Theorem 2.11 *The series representations for* Ψ_1 *and* Ψ_2 *in* (1.5)–(1.6) *satisfy the results*

$$\Psi_1(q^a, p^b; p^c, q^d; q, p, x, y) = \frac{(q^a; q)_\infty (p^b; p)_\infty}{(q^d; q)_\infty (p^c; p)_\infty} \sum_{r,s,k,\ell=0}^{\infty} \frac{(q^{a+k}; q)_\ell (q^{d-a}; q)_r (p^{c-b}; p)_s}{(q; q)_r (p; p)_s (q; q)_k (p; p)_\ell} \times q^{ar} p^{bs} \left(q^r x\right)^k \left(p^s y\right)^\ell, \tag{2.35}$$

$$\Psi_1(q^a, p^b; p^c, q^d; q, p, x, y) = \frac{(q^a; q)_\infty (p^b; p)_\infty (xq^a; q)_\infty}{(q^d; q)_\infty (p^c; p)_\infty (x; q)_\infty} \times \sum_{r,s,\ell=0}^{\infty} \frac{(q^{d-a}; q)_r (p^{c-b}; p)_s q^{ar} p^{bs}}{(q; q)_r (p; p)_s (xq^a; q)_\ell (p; p)_\ell} \left(q^r p^s y\right)^\ell, \tag{2.36}$$

$$\Psi_1(q^a, p^b; p^c, q^d; q, p, x, y) = \frac{(q^a; q)_\infty (p^b; p)_\infty}{(q^d; q)_\infty (p^c; p)_\infty} \sum_{r,s,\ell=0}^{\infty} \frac{(q^{d-a}; q)_r (p^{c-b}; p)_s q^{ar} p^{bs}}{(q; q)_r (p; p)_s (p; p)_\ell} \times \left(p^s y\right)^\ell {}_1\Phi_0(q^{a+\ell}; -; q, q^\ell x) \tag{2.37}$$

and

$$\Psi_2(q^a; p^b, q^c; q, p, x, y) = \frac{(q^a; q)_\infty}{(q^c; q)_\infty} \sum_{r,\ell,k=0}^{\infty} \frac{(q^{a+k}; q)_\ell (q^{c-a}; q)_r}{(p^b; p)_\ell (q; q)_r (q; q)_k (p; p)_\ell} q^{(a+k)r} x^k y^\ell. \tag{2.38}$$

Proof We can write the series of the function Ψ_1 as

$$\begin{aligned}
\Psi_1(q^a, p^b; p^c, q^d; q, p, x, y) &= \sum_{\ell,k=0}^{\infty} \frac{(q^a; q)_k (q^{a+k}; q)_\ell (p^b; p)_\ell}{(p^c; p)_\ell (q^d; q)_k (q; q)_k (p; p)_\ell} x^k y^\ell \\
&= \sum_{\ell,k=0}^{\infty} \frac{(q^a; q)_k (q^{a+k}; q)_\ell (p^b; p)_\infty (p^{c+\ell}; p)_\infty}{(p^{b+\ell}; p)_\infty (p^c; p)_\infty (q^d; q)_k (q; q)_k (p; p)_\ell} x^k y^\ell \\
&= \frac{(p^b; p)_\infty}{(p^c; p)_\infty} \sum_{\ell,k=0}^{\infty} \frac{(q^a; q)_k (q^{a+k}; q)_\ell (p^{c+\ell}; p)_\infty}{(p^{b+\ell}; p)_\infty (q^d; q)_k (q; q)_k (p; p)_\ell} x^k y^\ell \\
&= \frac{(q^a; q)_\infty (p^b; p)_\infty}{(q^d; q)_\infty (p^c; p)_\infty} \sum_{\ell,k=0}^{\infty} \frac{(q^{d+k}; q)_\infty (q^{a+k}; q)_\ell (p^{c+\ell}; p)_\infty}{(p^{b+\ell}; p)_\infty (q^{a+k}; q)_\infty (q; q)_k (p; p)_\ell} x^k y^\ell \\
&= \frac{(q^a; q)_\infty (p^b; p)_\infty}{(q^d; q)_\infty (p^c; p)_\infty} \sum_{\ell,k,s,r=0}^{\infty} \frac{(q^{a+k}; q)_\ell (q^{d-a}; q)_r (p^{c-b}; p)_s}{(q; q)_k (p; p)_\ell (q; q)_r (p; p)_s} \left(q^{a+k}\right)^r \left(p^{b+\ell}\right)^s x^k y^\ell.
\end{aligned}$$

The formulas (2.36)–(2.38) can be proved in a similar manner. □

Theorem 2.12 *The q-integral representations for* Ψ_1 *are true*

$$\Psi_1(q^a, p^b; p^c, q^d; q, p, x, y) = \frac{\Gamma_p(c)}{\Gamma_p(b)\Gamma_p(c-b)} \int_0^1 t^{b-1} \frac{(pt; p)_\infty}{(tp^{c-b}; p)_\infty} \times \Psi_1(q^a, p^e; p^e, q^d; q, p, x, yt) d_p t, \tag{2.39}$$

$$\Psi_1(q^a, p^b; p^c, q^d; q, p, x, y) = \frac{\Gamma_p(c)}{\Gamma_p(b)\Gamma_p(c-b)} \int_0^1 t^{b-1} \frac{(pt; p)_\infty}{(tp^{c-b}; p)_\infty} \times \Psi_2(q^a; 0, q^d; q, p, x, yt) d_p t, \tag{2.40}$$

$$\Psi_1(q^a, p^b; p^c, q^d; q, p, x, y) = \frac{1}{\Gamma_p(b)} \int_0^{\frac{1}{1-p}} E_p(-pt) t^{b-1} \times \Psi_2(q^a; p^c, q^d; q, p, x, (1-p)yt) d_p t \tag{2.41}$$

and

$$\Psi_1(q^a, p^b; p^c, q^d; q, p, x, y) = \frac{1}{\Gamma_q(a)} \int_0^{\frac{1}{1-q}} E_q(-qt)t^{a-1} {}_1\Phi_1(p^b; p^c; (1-q)yt) \times {}_0\Phi_1(-; q^d; (1-q)xt)d_qt. \tag{2.42}$$

Proof Using

$$\frac{(p^b; p)_\ell}{(p^c; p)_\ell} = \frac{\Gamma_p(c)}{\Gamma_p(b)\Gamma_p(c-b)} \int_0^1 t^{b+\ell-1} \frac{(pt; p)_\infty}{(tp^{c-b}; p)_\infty} d_pt,$$

for $0 < \Re(b) < \Re(c), c-b \neq 0, -1, -2, -3, \ldots, Re(b) > 0$ and $\ell \geq 0$, we obtain an p-integral representation for Ψ_1

$$\begin{aligned}
\Psi_1(q^a, p^b; p^c, q^d; q, p, x, y) &= \sum_{\ell,k=0}^{\infty} \frac{(q^a; q)_{k+\ell}(p^b; p)_\ell}{(p^c; p)_\ell(q^d; q)_k(q; q)_k(p; p)_\ell} x^k y^\ell \\
&= \frac{\Gamma_p(c)}{\Gamma_p(b)\Gamma_p(c-b)} \sum_{\ell,k=0}^{\infty} \frac{(q^a; q)_{k+\ell}}{(q^d; q)_k(q; q)_k(p; p)_\ell} x^k y^\ell \int_0^1 t^{b+\ell-1} \frac{(pt; p)_\infty}{(tp^{c-b}; p)_\infty} d_pt \\
&= \frac{\Gamma_p(c)}{\Gamma_p(b)\Gamma_p(c-b)} \int_0^1 \sum_{\ell,k=0}^{\infty} \frac{(q^a; q)_{k+\ell}(p^e; p)_\ell}{(p^e; p)_\ell(q^d; q)_k(q; q)_k(p; p)_\ell} x^k (yt)^\ell t^{b-1} \frac{(pt; p)_\infty}{(tp^{c-b}; p)_\infty} d_pt \\
&= \frac{\Gamma_p(c)}{\Gamma_p(b)\Gamma_p(c-b)} \int_0^1 t^{b-1} \Psi_1(q^a, p^e; p^e, q^d; q, p, x, yt) \frac{(pt; p)_\infty}{(tp^{c-b}; p)_\infty} d_pt.
\end{aligned}$$

Similarly, we obtain (2.40).

Using the p-shifted factorials $(p^b; p)_\ell$ and the p-Gamma functions are defined as follows (see [12,13])

$$(p^b; p)_\ell = \frac{(1-p)^\ell \Gamma_p(b+\ell)}{\Gamma_p(b)}$$

and

$$\Gamma_p(b) = \int_0^{\frac{1}{1-p}} E_p(-pt)t^{b-1} d_pt,$$

where $E_p(t)$ is the q-analogues of the exponential functions by

$$E_p(t) = \sum_{r=0}^{\infty} q^{\frac{r(r-1)}{2}} \frac{t^r}{[r]_q!},$$

we obtain (2.41)–(2.42). □

Theorem 2.13 *The summation formulas for* Ψ_1 *and* Ψ_2 *hold true*

$$\Psi_1(q^a, p^b; p^c, q^d; q, p, x, y) = \frac{1}{(x; q)_\infty} \sum_{\ell=0}^{\infty} \frac{(q^a; q)_\ell(p^b; p)_\ell}{(p^c; p)_\ell(p; p)_\ell} y^\ell {}_1\phi_1(q^{d-a-\ell}; q^d; q, xq^{a+\ell}), \tag{2.43}$$

$$\Psi_1(q^a, p^b; p^c, q^d; q, p, x, y) = \frac{1}{(q^d; q)_\infty(x; q)_\infty} \sum_{\ell=0}^{\infty} \frac{(q^a; q)_\ell(p^b; p)_\ell}{(p^c; p)_\ell(p; p)_\ell} (xq^{a+\ell}; q)_\infty \times y^\ell {}_1\phi_1(x; xq^{a+\ell}; q, q^d), \tag{2.44}$$

$$\Psi_2(q^a; p^b, q^c; q, p, x, y) = \frac{1}{(x; q)_\infty} \sum_{\ell=0}^{\infty} \frac{(q^a; q)_\ell}{(p^b; p)_\ell(p; p)_\ell} y^\ell {}_1\phi_1(q^{c-a-\ell}; q^c; q, xq^{a+\ell}) \tag{2.45}$$

and

$$\Psi_2(q^a; p^b, q^c; q, p, x, y) = \frac{1}{(q^c;q)_\infty (x;q)_\infty} \sum_{\ell=0}^{\infty} \frac{(q^a;q)_\ell}{(p^b;p)_\ell (p;p)_\ell} (xq^{a+\ell};q)_\infty \times y^\ell \; {}_1\phi_1(x; xq^{a+\ell}; q, q^c). \tag{2.46}$$

Proof The q-extension of kummer's transformation formulas for the basic confluent hypergeometric function is obtained (see [11,15])

$${}_2\phi_1(q^a, 0; q^c; q, x) = \frac{1}{(x;q)_\infty} \; {}_1\phi_1(q^{c-a}; q^c; q, xq^a) \tag{2.47}$$

and

$${}_2\phi_1(q^a, 0; q^c; q, x) = \frac{(xq^a;q)_\infty}{(q^c;q)_\infty (x;q)_\infty} \; {}_1\phi_1(x; xq^a; q, q^c). \tag{2.48}$$

Using (2.32) and (2.47), we obtain (2.43). Similarly, by using (2.33), (2.47) and (2.48), we can prove (2.44)–(2.46). □

Remark 2.1 The basic functions Ψ_1 and Ψ_2 are a q-analogue of the Humbert hypergeometric functions Ψ_1 and Ψ_2 defined by ([5], p. 225, Eqs. (23)–(24)):

$$\lim_{q\longrightarrow 1} \Psi_1\left(q^a, p^b; p^c, q^d; q, p, x, \frac{1}{1-q} y\right) = \Psi_1(a, b; c, d; x, y) \tag{2.49}$$

and

$$\lim_{q\longrightarrow 1} \Psi_2\left(q^a; p^b, q^c; q, p, x, \frac{1-p}{1-q} y\right) = \Psi_2(a, b; c, d; x, y). \tag{2.50}$$

Proof Using the limit

$$\lim_{q\longrightarrow 1} \frac{(q^a;q)_n}{(1-q)^n} = (a)_n,$$

we obtain (2.49)–(2.50). □

Theorem 2.14 *The relationship holds true between the bibasic functions Ψ_1 and Ψ_2:*

$$\lim_{b\longrightarrow\infty} \Psi_1(q^a, p^b; p^c, q^d; q, p, x, y) = \Psi_2(q^a; p^c, q^d; q, p, x, y). \tag{2.51}$$

Proof Using the limit

$$\lim_{b\longrightarrow\infty} (p^b; p)_n = (0; p)_n = 1,$$

we obtain (2.51). □

2.1 Particular cases

Here, we discuss and give theorems which are the following particular cases of Theorem 2.10:

Theorem 2.15 *The relationships between for Ψ_1, Ψ_2 and basic hypergeometric functions hold true*

$$\Psi_1(q^a, q^b; q^c, q^d; q, 0, y) = {}_2\phi_1(q^a, q^b; q^c; q, y), \tag{2.52}$$

$$\Psi_1(q^a, q^b; q^c, q^d; q, x, 0) = {}_2\phi_1(q^a, 0; q^d; q, x) \tag{2.53}$$

and

$$\begin{aligned} \Psi_2(q^a; q^b, q^c; q, x, 0) &= {}_1\phi_0(q^a; -; q, x), \\ \Psi_2(q^a; q^b, q^c; q, 0, y) &= {}_2\phi_1(q^a, 0; q^b; q, y). \end{aligned} \tag{2.54}$$

Proof Set $p = q$ in (2.32), we obtain

$$\Psi_1(q^a, q^b; q^c, q^d; q, 0, y) = {}_2\phi_1(q^a, q^b; q^c; q, y).$$

The identities (2.53)–(2.54) can be proved in a like manner. □

Theorem 2.16 *The summation formulas for* Ψ_1 *and* Ψ_2 *hold true:*

$$\begin{aligned}\Psi_1(q^a, q^b; q^c, q^d; q, x, y) =& \frac{(q^a; q)_\infty (q^b; q)_\infty (q^a y; q)_\infty}{(q^c; q)_\infty (q^d; q)_\infty (y; q)_\infty} \sum_{r,s=0}^{\infty} \frac{(q^{d-a}; q)_r (q^{c-b}; q)_s (y; q)_s}{(q^a y; q)_s (q; q)_s (q; q)_r} \\ & \times q^{ar} q^{bs}\ {}_2\phi_1(0, 0; q^{a+s} y; q, q^r x), \\ \Psi_1(q^a, q^b; q^c, q^d; q, x, y) =& \frac{(q^a; q)_\infty (q^b; q)_\infty (x q^a; q)_\infty}{(q^c; q)_\infty (q^d; q)_\infty (x; q)_\infty} \sum_{r,s=0}^{\infty} \frac{(q^{d-a}; q)_r (q^{c-b}; q)_s}{(q; q)_s (q; q)_r} \\ & \times q^{ar} q^{bs}\ {}_2\phi_1(0, 0; x q^a; q, q^r q^s y), \\ \Psi_1(q^a, q^b; q^c, q^d; q, x, y) =& \frac{(q^a; q)_\infty (q^b; q)_\infty}{(q^c; q)_\infty (q^d; q)_\infty} \sum_{r,s,\ell=0}^{\infty} \frac{(q^{d-a}; q)_r (q^{c-b}; q)_s}{(q; q)_s (q; q)_r (q; q)_\ell} \\ & \times q^{ar} q^{bs} \Big(q^s y\Big)^{\ell}\ {}_1\Phi_0(q^{a+\ell}; -; q, q^{\ell} x)\end{aligned} \tag{2.55, 2.56}$$

and

$$\begin{aligned}\Psi_2(q^a; q^b, q^c; q, x, y) &= \frac{(q^a; q)_\infty}{(q^b; q)_\infty} \sum_{\ell=0}^{\infty} \frac{(q^{b+\ell}; q)_\infty}{(q^{a+\ell}; q)_\infty (q; q)_\ell} y^{\ell}\ {}_2\phi_1(q^{a+\ell}, 0; q^c; q, x), \\ \Psi_2(q^a; q^b, q^c; q, x, y) &= \frac{(q^a; q)_\infty}{(q^c; q)_\infty} \sum_{k=0}^{\infty} \frac{(q^{c+k}; q)_\infty}{(q^{a+k}; q)_\infty (q; q)_k} x^{k}\ {}_2\phi_1(q^{a+k}, 0; q^b; q, y).\end{aligned} \tag{2.57}$$

Proof Take $p = q$ with a similar argument as in Theorem 2.12 and use to transform the series on the right, simplification gives the required results (2.55)–(2.57). □

Corollary 2.1 *The summation formula for* Ψ_1 *hold true*

$$\begin{aligned}\Psi_1(q^a, q^b; q^c, q^d; q, x, y) =& \frac{(q^a; q)_\infty (q^b; q)_\infty (q^a y; q)_\infty}{(q^d; q)_\infty (q^c; q)_\infty (y; q)_\infty} \sum_{r,s=0}^{\infty} \frac{(q^{d-a}; q)_r (q^{c-b}; q)_s (y; q)_s}{(q^a y; q)_s (q; q)_s (q; q)_r} \\ & \times \frac{q^{ar+bs}}{(q^{a+s} y; q)_\infty (x q^r; q)_\infty}\ {}_1\phi_1(x q^r; 0; q, q^{a+s} y), \\ \Psi_1(q^a, q^b; q^c, q^d; q, x, y) =& \frac{(q^a; q)_\infty (q^b; q)_\infty (q^a y; q)_\infty}{(q^d; q)_\infty (q^c; q)_\infty (y; q)_\infty} \sum_{r,s=0}^{\infty} \frac{(q^{d-a}; q)_r (q^{c-b}; q)_s (y; q)_s}{(q^a y; q)_s (q; q)_s (q; q)_r} \\ & \times \frac{q^{ar+bs}}{(x q^r; q)_\infty}\ {}_0\phi_1(-; q^{a+s} y; q, x q^r q^{a+s} y),\end{aligned} \tag{2.58}$$

and

$$\begin{aligned}\Psi_1(q^a, q^b; q^c, q^d; q, x, y) =& \frac{(q^a; q)_\infty (q^b; q)_\infty (x q^a; q)_\infty}{(q^d; q)_\infty (q^c; q)_\infty (x; q)_\infty} \sum_{r,s=0}^{\infty} \frac{(q^{d-a}; q)_r (q^{c-b}; q)_s}{(q; q)_s (q; q)_r} \\ & \times \frac{q^{ar+bs}}{(x q^a; q)_\infty (q^{r+s} y; q)_\infty}\ {}_1\phi_1(q^{r+s} y; 0; q, x q^a), \\ \Psi_1(q^a, q^b; q^c, q^d; q, x, y) =& \frac{(q^a; q)_\infty (q^b; q)_\infty (x q^a; q)_\infty}{(q^d; q)_\infty (q^c; q)_\infty (x; q)_\infty} \sum_{r,s=0}^{\infty} \frac{(q^{d-a}; q)_r (q^{c-b}; q)_s}{(q; q)_s (q; q)_r} \\ & \times \frac{q^{ar+bs}}{(q^{r+s} y; q)_\infty}\ {}_0\phi_1(-; x q^a; q, x y q^{a+r+s}).\end{aligned} \tag{2.59}$$

Proof Using the Heine's transformations formulas for basic hypergeometric functions: (see [11])

$$\begin{aligned} {}_2\phi_1(0, 0; q^c; q, x) &= \frac{1}{(q^c; q)_\infty (x; q)_\infty} {}_1\phi_1(x; 0; q, q^c) \\ &= \frac{1}{(x; q)_\infty} {}_0\phi_1(-; q^c; q, xq^c), \end{aligned} \tag{2.60}$$

and replacing q^c and x by $q^{a+s}y$ and xq^r in (2.60), we get (2.58), replacing q^c and x by xq^a and $q^{r+s}y$ in (2.60), we obtain (2.59). □

3 Concluding remarks

It may be remarked, finally, by using the same technique in above by new definitions of bibasic Humbert hypergeometric functions Ψ_1 and Ψ_2 for the numerator and denominator parameters with a few more new and elegant ones on quantum analogue. Furthermore, we obtained interesting results. We believe that a detailed study of the bibasic Humbert hypergeometric functions on properties for this functions should be very interesting to permit elegant and neat extensions and hence have not been included herein.

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