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On separation axioms in fuzzifying topology

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Abstract

In the present paper we introduce R_0- and R_1-separation axioms in fuzzifying topology and study their relations with T_1- and T_2-separation axioms, respectively. Furthermore, we introduce and study *semi-*T_0-, *semi-*R_0-, *semi-*T_1-, *semi-*R_1-, *semi-*T_2(semi-Hausdorff)-, *semi-*T_3(semi-regularity)- and *semi-*T_4(semi-normality)-separation axioms in fuzzifying topology and give some of their characterizations as well as the relations of these axioms and other separation axioms in fuzzifying topology introduced by Shen, Fuzzy Sets and Systems 57 (1993) 111–123. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction and preliminaries

Chang [1], Wong [8], Lowen [5], Hutton [2], Pu and Liu [6], etc., discussed various aspects of fuzzy topology with crisp methods. Ying [9,10] introduced fuzzifying topology and elementarily developed fuzzy topology from a new direction with the semantic method of continuous valued logic. In the framework of fuzzifying topology, Shen [7] introduced and studied T_0-, T_1-, T_2(Hausdorff)-, T_3(regularity)- and T_4(normality)-separation axioms in fuzzifying topology. In [4], the authors introduced and studied the concepts of the family of fuzzifying semi-open sets, fuzzifying semi-neighborhood structure of a point and fuzzifying semi-closure. In the present paper, we add the R_0- and R_1-separation axioms and study their relations with the T_1- and T_2-separation axioms, respectively. Also, in fuzzifying topology we introduce and study *semi-*T_0-, *semi-*R_0-, *semi-*T_1-, *semi-*R_1-, *semi-*T_2(semi-Hausdorff)-, *semi-*T_3(semi-regularity)- and *semi-*T_4(semi-normality)-separation axioms in fuzzifying topology.

The reader is assumed to be familiar with Ying's papers [9,10].

First, we display the fuzzy logical and corresponding set-theoretical notations used in this paper.

For any formula φ, the symbol $[\varphi]$ means the truth value of φ, where the set of truth values is the unit interval $[0, 1]$. A formula φ is valid, we write $\models \varphi$, if and only if $[\varphi] = 1$ for every interpretation.

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(1) $[\alpha] := \alpha(\alpha \in [0,1])$; $[\varphi \wedge \psi] := \min([\varphi], [\psi])$,

$$[\varphi \to \psi] := \min(1, 1 - [\varphi] + [\psi]).$$

(2) If $\tilde{A} \in \mathscr{I}(X)$, where $\mathscr{I}(X)$ is the family of fuzzy sets of X, then

$$[x \in \tilde{A}] := \tilde{A}(x).$$

(3) If X is the universe of discourse, then

$$[\forall x \varphi(x)] := \inf_{x \in X} [\varphi(x)].$$

In addition, the following derived formulae are given:
(1) $[\neg\varphi] := [\varphi \to 0] = 1 - [\varphi]$,
(2) $[\varphi \vee \psi] := [\neg(\neg\varphi \vee \neg\psi)] = \max([\varphi], [\psi])$,
(3) $[\varphi \leftrightarrow \psi] := [(\varphi \to \psi) \wedge (\psi \to \varphi)]$,
(4) $[\varphi \mathbin{\dot\wedge} \psi] := [\neg(\varphi \to \neg\psi)] = \max(0, [\varphi] + [\psi] - 1)$,
(5) $[\varphi \mathbin{\dot\vee} \psi] := [\neg\varphi \to \psi] = [\neg(\neg\varphi \mathbin{\dot\wedge} \neg\psi)] = \min(1, [\varphi] + [\psi])$,
(6) $[\exists x \varphi(x)] := [\neg\forall x \neg\varphi(x)] = \sup_{x \in X} [\varphi(x)]$,
(7) If $\tilde{A}, \tilde{B} \in \mathscr{I}(X)$, then
(a) $[\tilde{A} \subseteq \tilde{B}] := [\forall x(x \in \tilde{A} \to x \in \tilde{B})] = \inf_{x \in X} \min(1, 1 - \tilde{A}(x) + \tilde{B}(x))$,
(b) $\tilde{A} \equiv \tilde{B} := [(\tilde{A} \subseteq \tilde{B}) \wedge (\tilde{B} \subseteq \tilde{A})]$.

Second, we give some definitions and results in fuzzifying topology.

Definition 1.1. Let X be the universe of discourse and let $\tau \in \mathscr{I}(P(X))$, where $P(X)$ is the power set of X, satisfying the following conditions:
(1) $\models X \in \tau$,
(2) for any $A, B \in P(X)$, $\models (A \in \tau) \wedge (B \in \tau) \to A \cap B \in \tau$,
(3) for any $\{A_\lambda : \lambda \in \Lambda\} \subseteq P(X)$, $\models \forall \lambda(\lambda \in \Lambda \to A_\lambda \in \tau) \to \bigcup_{\lambda \in \Lambda} A_\lambda \in \tau$.

Then τ is called a fuzzifying topology and (X, τ) is called a fuzzifying topological space. The family of all fuzzifying closed sets will be denoted by F_τ, or if there is no confusion by F, and defined as follows:

$$A \in F := X \sim A \in \tau,$$

where $X \sim A$ is the complement of A.

Definition 1.2. Let (X, τ) be a fuzzifying topological space.

(1) The fuzzifying neighborhood system of a point $x \in X$ is denoted by $N_x \in \mathscr{I}(P(X))$ and defined as follows:

$$N_x(A) = \sup_{x \in B \subseteq A} \tau(B).$$

(2) The interior of a set $A \in P(X)$ is denoted by $A^\circ \in \mathscr{I}(X)$ and defined as follows:

$$A^\circ(x) := N_x(A).$$

(3) The closure of a set $A \in P(X)$ is denoted by $\bar{A} \in \mathscr{I}(X)$ and defined as follows:

$$\bar{A}(x) = 1 - N_x(X \sim A).$$

(4) $\beta \in \mathscr{I}(P(X))$ is a base of τ if and only if $\tau = \beta^{(U)}$, i.e.,

$$\tau(A) = \sup_{\bigcup_{\lambda \in \Lambda} B_\lambda = A} \bigwedge_{\lambda \in \Lambda} \beta(B_\lambda) \quad \text{(Theorem 4.1 [9])}.$$

(5) $\varphi \in \mathscr{I}(P(X))$ is a sub base of τ if $\varphi^{\sqcap}$ is a base of τ, i.e.,

$$\tau(A) = \sup_{\bigcup_{\lambda \in \Lambda} D_\lambda = A} \inf_{\lambda \in \Lambda} \sup_{\bigcap_{\lambda_i \in I_\lambda} D_{\lambda_i} = D_\lambda} \inf_{\lambda_i \in I_\lambda} \varphi(D_{\lambda_i}).$$

Before we recall the definitions of some separation axioms in fuzzifying topology introduced by Shen [7] we introduce the following remark.

Remark 1.1. For simplicity we put the following notations:

$$K_{x,y} := \exists A((A \in N_x \wedge y \notin A) \vee (A \in N_y \wedge x \notin A)),$$

$$H_{x,y} := \exists B\, \exists C((B \in N_x \wedge y \notin B) \wedge (C \in N_y \wedge x \notin C)),$$

$$M_{x,y} := \exists B\, \exists C(B \in N_x \wedge C \in N_y \wedge B \cap C = \emptyset),$$

$$V_{x,D} := \exists A\, \exists B(A \in N_x \wedge B \in \tau \wedge D \subseteq B \wedge A \cap B = \emptyset),$$

$$W_{A,B} := \exists G\, \exists H(G \in \tau \wedge H \in \tau \wedge A \subseteq G \wedge B \subseteq H \wedge G \cap H = \emptyset).$$

Definition 1.3. Let Ω be the class of all fuzzifying topological spaces. The unary fuzzy predicates $T_i \in \mathscr{I}(\Omega)$, $i = 0, 1, 2, 3, 4$, are defined as follows:

$$(X, \tau) \in T_0 := \forall x \forall y(x \in X \wedge y \in X \wedge x \neq y) \rightarrow K_{x,y},$$

$$(X, \tau) \in T_1 := \forall x \forall y(x \in X \wedge y \in X \wedge x \neq y) \rightarrow H_{x,y},$$

$$(X, \tau) \in T_2 := \forall x \forall y(x \in X \wedge y \in X \wedge x \neq y) \rightarrow M_{x,y},$$

$$(X, \tau) \in T_3 := \forall x \forall D(x \in X \wedge D \in F \wedge x \notin D) \rightarrow V_{x,D},$$

$$(X, \tau) \in T_4 := \forall A \forall B(A \in F \wedge B \in F \wedge A \cap B = \emptyset) \rightarrow W_{A,B}.$$

For any fuzzifying topological space (X, τ),

$$\models T_1(X, \tau) \leftrightarrow \forall x(\{x\} \in F).$$

Finally, we recall some definitions and results from [4] which are useful in the sequel.

Definition 1.4. Let (X, τ) be a fuzzifying topological space.

(1) The family of fuzzifying semi-open sets is denoted by $S\tau \in \mathscr{I}(P(X))$ and defined as

$$S\tau(A) = \inf_{x \in A} A^{\circ -}(x).$$

(2) The family of fuzzifying semi-closed sets is denoted by $SF \in \mathscr{I}(P(X))$ and defined as follows:

$$SF(A) = S\tau(X \sim A).$$

(3) The semi-neighborhood system of a point $x\in X$ is denoted by $SN_x\in\mathscr{I}(P(X))$ and defined as follows:

$$SN_x(A)=\sup_{x\in B\subseteq A} S\tau(B).$$

(4) The semi-closure of a set $A\in P(X)$ is denoted by *semi-cl*$(A)\in\mathscr{I}(X)$ and defined as follows:

$$\textit{semi-cl}(A)(x)=1-SN_x(X\sim A).$$

Theorem 1.1. *Let* (X,τ) *be a fuzzifying topological space. Then,*
(1) $\models \tau\subseteq S\tau$,
(2) $\models F\subseteq SF$.

Theorem 1.2. *The mapping* $SN: X\to\mathscr{I}^N(P(X))$, $x\mapsto SN_x$, *where* $\mathscr{I}^N(P(X))$ *is the set of all normal fuzzy subsets of* $P(X)$ *has the following properties*:
(1) $\models A\in SN_x\to x\in A$,
(2) $\models A\subseteq B\to (A\in SN_x\to B\in SN_x)$,
(3) $\models A\in SN_x\to \exists H(H\in SN_x\wedge H\subseteq A\wedge \forall y(y\in H\to H\in SN_y))$.

Theorem 1.3.

$$S\tau(A)=\inf_{x\in A}\sup_{x\in B\subseteq A} S\tau(B).$$

Corollary 1.3.

$$S\tau(A)=\inf_{x\in A} SN_x(A).$$

2. R_0- and R_1-separation axioms

Definition 2.1. Let Ω be the class of all fuzzifying topological spaces. The unary fuzzy predicates $R_0,R_1\in\mathscr{I}(\Omega)$ are defined as follows:

$$(X,\tau)\in R_0:=\forall x\forall y(x\in X\wedge y\in X\wedge x\neq y)\to(K_{x,y}\to H_{x,y}),$$
$$(X,\tau)\in R_1:=\forall x\forall y(x\in X\wedge y\in X\wedge x\neq y)\to(K_{x,y}\to M_{x,y}).$$

Lemma 2.1. (1) $\models M_{x,y}\to H_{x,y}$,
(2) $\models H_{x,y}\to K_{x,y}$,
(3) $\models M_{x,y}\to K_{x,y}$.

Proof. (1) Since $\{B,C\in P(X): B\cap C=\phi\}\subseteq\{B,C\in P(X): y\notin B\wedge x\notin C\}$, then $[M_{x,y}]=\sup_{B\cap C=\phi}\min(N_x(B), N_y(C))\leqslant \sup_{y\notin B,x\notin C}\min(N_x(B),N_y(C))=[H_{x,y}]$.
(2) $[K_{x,y}]=\max(\sup_{y\notin A}N_x(A),\sup_{x\notin A}N_y(A))\geqslant \sup_{y\notin A}N_x(A)\geqslant \sup_{y\notin A,x\notin B}(N_x(A)\wedge N_y(B))=[H_{x,y}]$.
(3) From (1) and (2) it is obvious. □

Theorem 2.1.

$$\models (X,\tau)\in R_1\to (X,\tau)\in R_0.$$

Proof. From Lemma 2.1(1) we have,

$$R_0(X,\tau) = \inf_{x\neq y} \min(1, 1-[K_{x,y}]+[H_{x,y}]) \geqslant \inf_{x\neq y} \min(1, 1-[K_{x,y}]+[M_{x,y}]) = R_1(X,\tau). \qquad \square$$

Lemma 2.2. *For any* $\alpha, \beta \in I$,

$$\models \alpha \to (\beta \to \alpha).$$

Proof.

$$[\alpha \to (\beta \to \alpha)] = \min(1, 1-\alpha+\min(1, 1-\beta+\alpha)) = 1. \qquad \square$$

Theorem 2.2. (1) $\models (X,\tau)\in T_1 \to (X,\tau)\in R_0$.
(2) $\models (X,\tau)\in T_1 \to (X,\tau)\in R_0 \wedge (X,\tau)\in T_0$,
(3) *If* $T_0(X,\tau)=1$, *then*

$$\models (X,\tau)\in T_1 \leftrightarrow (X,\tau)\in R_0 \wedge (X,\tau)\in T_0.$$

Proof. (1) Applying Lemma 2.2 we have

$$T_1(X,\tau) = \inf_{x\neq y}[H_{x,y}] \leqslant \inf_{x\neq y}[K_{x,y} \to H_{x,y}] = R_0(X,\tau).$$

(2) It is obtained from (1) and since, $\models (X,\tau)\in T_1 \to (X,\tau)\in T_0$ [7].
(3) Since $T_0(X,\tau)=1$, then for every $x, y\in X$ such that $x\neq y$ we have $[K_{x,y}]=1$. Now,

$$\begin{aligned} R_0(X,\tau)\wedge T_0(X,\tau) &= R_0(X,\tau) \\ &= \inf_{x\neq y} \min(1, 1-[K_{x,y}]+[H_{x,y}]) = \inf_{x\neq y}[H_{x,y}] = T_1(X,\tau). \qquad \square \end{aligned}$$

Theorem 2.3. (1) $\models (X,\tau)\in R_0 \mathbin{\dot\wedge} (X,\tau)\in T_0 \to (X,\tau)\in T_1$,
(2) *If* $T_0(X,\tau)=1$, *then*

$$\models (X,\tau)\in R_0 \mathbin{\dot\wedge} (X,\tau)\in T_0 \leftrightarrow (X,\tau)\in T_1.$$

Proof. (1)

$$\begin{aligned} &[(X,\tau)\in R_0 \mathbin{\dot\wedge} (X,\tau)\in T_0] \\ &\quad = \max(0, R_0(X,\tau)+T_0(X,\tau)-1) \\ &\quad = \max\left(0, \inf_{x\neq y} \min(1, 1-[K_{x,y}]+[H_{x,y}]) + \inf_{x\neq y}[K_{x,y}]-1\right) \\ &\quad \leqslant \max\left(0, \inf_{x\neq y}(\min(1, 1-[K_{x,y}]+[H_{x,y}])+[K_{x,y}]-1)\right) \\ &\quad = \max\left(0, \inf_{x\neq y}(1-[K_{x,y}]+[H_{x,y}]+[K_{x,y}]-1)\right) = \inf_{x\neq y}[H_{x,y}] = T_1(X,\tau). \end{aligned}$$

(2)

$$[(X,\tau)\in R_0 \mathbin{\dot\wedge} (X,\tau)\in T_0] = R_0(X,\tau)$$
$$= \inf_{x\neq y} \min(1,\ 1-[K_{x,y}]+[H_{x,y}]) = \inf_{x\neq y}[H_{x,y}] = T_1(X,\tau),$$

since $T_0(X,\tau)=1$, for each $x,y\in X$ such that $x\neq y$ we have $[K_{x,y}]=1$. □

Theorem 2.4. (1) $\models (X,\tau)\in T_0 \to ((X,\tau)\in R_0 \to (X,\tau)\in T_1)$,
(2) $\models (X,\tau)\in R_0 \to ((X,\tau)\in T_0 \to (X,\tau)\in T_1)$.

Proof. (1) From Theorems 2.2(1) and 2.3(1) we have,

$$[(X,\tau)\in T_0 \to ((X,\tau)\in R_0 \to (X,\tau)\in T_1)]$$
$$= \min(1, 1-[(X,\tau)\in T_0] + \min(1, 1-[(X,\tau)\in R_0] + [(X,\tau)\in T_1]))$$
$$= \min(1, 1-[(X,\tau)\in T_0] + 1-[(X,\tau)\in R_0] + [(X,\tau)\in T_1])$$
$$= \min(1, 1-([(X,\tau)\in R_0] + [(X,\tau)\in T_0]-1) + [(X,\tau)\in T_1]) = 1.$$

(2)

$$[(X,\tau)\in R_0 \to ((X,\tau)\in T_0 \to (X,\tau)\in T_1)]$$
$$= \min(1, 1-[(X,\tau)\in R_0] + \min(1, 1-[(X,\tau)\in T_0] + [(X,\tau)\in T_1]))$$
$$= \min(1, 1-[(X,\tau)\in R_0] + 1-[(X,\tau)\in T_0] + [(X,\tau)\in T_1])$$
$$= \min(1, 1-([(X,\tau)\in R_0] + [(X,\tau)\in T_0]-1) + [(X,\tau)\in T_1]) = 1. \qquad \square$$

Theorem 2.5. (1) $\models (X,\tau)\in T_2 \to (X,\tau)\in R_1$,
(2) $\models (X,\tau)\in T_2 \to (X,\tau)\in R_1 \wedge (X,\tau)\in T_0$,
(3) *If* $T_0(X,\tau)=1$, *then*

$$\models (X,\tau)\in T_2 \leftrightarrow (X,\tau)\in R_1 \wedge (X,\tau)\in T_0.$$

Proof. (1) Applying Lemma 2.2 we have,

$$T_2(X,\tau) = \inf_{x\neq y}[M_{x,y}] \leqslant \inf_{x\neq y}[K_{x,y} \to M_{x,y}] = R_1(X,\tau).$$

(2) It is obtained from (1) and since, $\models (X,\tau)\in T_2 \to (X,\tau)\in T_0$ [7].
(3) Since $T_0(X,\tau)=1$, then for each $x,y\in X$ such that $x\neq y$ we have $[K_{x,y}]=1$. Now,

$$R_1(X,\tau)\wedge T_0(X,\tau) = R_1(X,\tau)$$
$$= \inf_{x\neq y} \min(1, 1-[K_{x,y}] + [M_{x,y}])$$
$$= \inf_{x\neq y} [M_{x,y}] = T_2(X,\tau). \qquad \square$$

Remark 2.1. In the crisp setting, i.e., if the underlying fuzzifying topology is the ordinary topology one can have that
(1) $\models (X,\tau)\in R_0 \wedge (X,\tau)\in T_0 \leftrightarrow (X,\tau)\in T_1$,
(2) $\models (X,\tau)\in R_1 \wedge (X,\tau)\in T_0 \leftrightarrow (X,\tau)\in T_2$,

but these statements may not be true in general in fuzzifying topology as illustrated by the following counter-example.

Counterexample 2.1. Let $X=\{x,y\}$ and let τ be a fuzzifying topology on X defined as follows: $\tau(X)=\tau(\phi)=1$, $\tau(\{x\})=\frac{1}{4}$ and $\tau(\{y\})=\frac{1}{5}$. One can have that $T_0(X,\tau)=\frac{1}{4}$, $R_0(X,\tau)=R_1(X,\tau)=\frac{19}{20}$ and $T_1(X,\tau)=T_2(X,\tau)=\frac{1}{5}$. Hence,

$$R_0(X,\tau)\wedge T_0(X,\tau)=\tfrac{19}{20}\wedge\tfrac{1}{4}=\tfrac{1}{4}\neq\tfrac{1}{5}=T_1(X,\tau),\qquad R_1(X,\tau)\wedge T_0(X,\tau)=\tfrac{19}{20}\wedge\tfrac{1}{4}=\tfrac{1}{4}\neq\tfrac{1}{5}=T_2(X,\tau).$$

Theorem 2.6. (1) $\models (X,\tau)\in R_1 \mathbin{\dot\wedge} (X,\tau)\in T_0 \to (X,\tau)\in T_2$,
(2) *If* $T_0(X,\tau)=1$, *then*

$$\models (X,\tau)\in R_1 \mathbin{\dot\wedge} (X,\tau)\in T_0 \leftrightarrow (X,\tau)\in T_2.$$

Proof. (1)

$$\begin{aligned}[(X,\tau)\in R_1 \mathbin{\dot\wedge} (X,\tau)\in T_0] &= \max(0,R_1(X,\tau)+T_0(X,\tau)-1)\\ &= \max\left(0,\inf_{x\neq y}\min(1,1-[K_{x,y}]+[M_{x,y}])+\inf_{x\neq y}[K_{x,y}]-1\right)\\ &\leqslant \max\left(0,\inf_{x\neq y}(\min(1,1-[K_{x,y}]+[M_{x,y}])+[K_{x,y}]-1)\right)\\ &= \inf_{x\neq y}[M_{x,y}]=T_2(X,\tau).\end{aligned}$$

(2)

$$\begin{aligned}[(X,\tau)\in R_1 \mathbin{\dot\wedge} (X,\tau)\in T_0] &= R_1(X,\tau)\\ &= \inf_{x\neq y}\min(1,1-[K_{x,y}]+[M_{x,y}])=\inf_{x\neq y}[M_{x,y}]=T_2(X,\tau),\end{aligned}$$

because $T_0(X,\tau)=1$, we have for each $x,y\in X$ such that $x\neq y$ that $[K_{x,y}]=1$. □

Theorem 2.7. (1) $\models (X,\tau)\in T_0\to((X,\tau)\in R_1\to(X,\tau)\in T_2)$,
(2) $\models (X,\tau)\in R_1\to((X,\tau)\in T_0\to(X,\tau)\in T_2)$.

Proof. (1) From Theorems 2.5(1) and 2.6(1) we have,

$$\begin{aligned}&[(X,\tau)\in T_0\to((X,\tau)\in R_1\to(X,\tau)\in T_2)]\\ &\quad=\min(1,1-[(X,\tau)\in T_0]+\min(1,1-[(X,\tau)\in R_1]+[(X,\tau)\in T_2]))\\ &\quad=\min(1,1-[(X,\tau)\in T_0]+1-[(X,\tau)\in R_1]+[(X,\tau)\in T_2])\\ &\quad=\min(1,1-[(X,\tau)\in T_0]+[(X,\tau)\in R_1]-1)+[(X,\tau)\in T_2])=1.\end{aligned}$$

(2)

$$\begin{aligned}&[(X,\tau)\in R_1\to((X,\tau)\in T_0\to(X,\tau)\in T_2)]\\ &\quad=\min(1,1-[(X,\tau)\in R_1]+\min(1,1-[(X,\tau)\in T_0]+[(X,\tau)\in T_2]))\end{aligned}$$

$$= \min(1, 1 - [(X,\tau) \in R_1] + 1 - [(X,\tau) \in T_0] + [(X,\tau) \in T_2])$$
$$= \min(1, 1 - ([(X,\tau) \in R_1] + [(X,\tau) \in T_0] - 1) + [(X,\tau) \in T_2]) = 1,$$

because $\models (X,\tau) \in T_0 \to (X,\tau) \in T_2$ [7] and applying Theorem 2.6(1). □

3. Fuzzifying semi-separation axioms

Remark 3.1. For simplicity we put the following notations:

$$SK_{x,y} := \exists A((A \in SN_x \wedge y \notin A) \vee (A \in SN_y \wedge x \notin A)),$$

$$SH_{x,y} := \exists B\, \exists C((B \in SN_x \wedge y \notin B) \wedge (C \in SN_y \wedge x \notin C)),$$

$$SM_{x,y} := \exists B\, \exists C(B \in SN_x \wedge C \in SN_y \wedge B \cap C = \emptyset),$$

$$SV_{x,D} := \exists A\, \exists B(A \in SN_x \wedge B \in S\tau \wedge D \subseteq B \wedge A \cap B = \emptyset),$$

$$SW_{A,B} := \exists G\, \exists H(G \in S\tau \wedge H \in S\tau \wedge A \subseteq G \wedge B \subseteq H \wedge G \cap H = \emptyset).$$

Definition 3.1. Let Ω be the class of all fuzzifying topological spaces. The unary fuzzy predicates *semi*-$T_i \in \mathscr{I}(\Omega)$, $i = 0,1,2,3,4$ and *semi*-$R_i \in \mathscr{I}(\Omega)$, $i = 0,1$ are defined as follows:

$$(X,\tau) \in semi\text{-}T_0 := \forall x\, \forall y\, (x \in X \wedge y \in X \wedge x \neq y) \to SK_{x,y},$$

$$(X,\tau) \in semi\text{-}T_1 := \forall x\, \forall y\, (x \in X \wedge y \in X \wedge x \neq y) \to SH_{x,y},$$

$$(X,\tau) \in semi\text{-}T_2 := \forall x\, \forall y\, (x \in X \wedge y \in X \wedge x \neq y) \to SM_{x,y},$$

$$(X,\tau) \in semi\text{-}T_3 := \forall x\, \forall D\, (x \in X \wedge D \in F \wedge x \notin D) \to SV_{x,D},$$

$$(X,\tau) \in semi\text{-}T_4 := \forall A\, \forall B\, (A \in F \wedge B \in F \wedge A \cap B = \emptyset) \to SW_{A,B},$$

$$(X,\tau) \in semi\text{-}R_0 := \forall x\, \forall y\, (x \in X \wedge y \in X \wedge x \neq y) \to (SK_{x,y} \to SH_{x,y}),$$

$$(X,\tau) \in semi\text{-}R_1 := \forall x\, \forall y\, (x \in X \wedge y \in X \wedge x \neq y) \to (SK_{x,y} \to SM_{x,y}).$$

Lemma 3.1. (1) $\models K_{x,y} \to SK_{x,y}$,
(2) $\models H_{x,y} \to SH_{x,y}$,
(3) $\models M_{x,y} \to SM_{x,y}$,
(4) $\models V_{x,D} \to SV_{x,D}$,
(5) $\models W_{A,B} \to SW_{A,B}$.

Proof. From Theorem 1.1(1), $\models \tau \subseteq S\tau$ and so one can deduce that $N_x(A) \leqslant SN_x(A)$ for any $A \in P(X)$, the proof is immediate. □

Theorem 3.1.

$$\models (X,\tau) \in T_i \to (X,\tau) \in semi\text{-}T_i,$$

where $i = 0,1,2,3,4$.

Proof. It is obtained from Lemma 3.1.

Theorem 3.2. *If* $T_0(X,\tau)=1$, *then* (1) $\models (X,\tau)\in R_0 \to (X,\tau)\in semi\text{-}R_0$,
(2) $\models (X,\tau)\in R_1 \to (X,\tau)\in semi\text{-}R_1$.

Proof. Since $T_0(X,\tau)=1$ then for each $x,y\in X$ and $x\neq y$ we have, $[K_{x,y}]=1$ and so, $[SK_{x,y}]=1$.
(1) $R_0(X,\tau)=\inf_{x\neq y}[K_{x,y}\to H_{x,y}]\leqslant \inf_{x\neq y}[K_{x,y}\to SH_{x,y}]=\inf_{x\neq y}[SK_{x,y}\to SH_{x,y}]=semi\text{-}R_0(X,\tau)$.
(2) $R_1(X,\tau)=\inf_{x\neq y}[K_{x,y}\to M_{x,y}]\leqslant \inf_{x\neq y}[K_{x,y}\to SM_{x,y}]=\inf_{x\neq y}[SK_{x,y}\to SM_{x,y}]=semi\text{-}R_1(X,\tau)$.

Lemma 3.2. (1) $\models SM_{x,y}\to SH_{x,y}$,
(2) $\models SH_{x,y}\to SK_{x,y}$,
(3) $\models SM_{x,y}\to SK_{x,y}$.

Proof. The proof is similar to the proof of Lemma 2.1. □

Theorem 3.3. (1) $\models (X,\tau)\in semi\text{-}T_1 \to (X,\tau)\in semi\text{-}T_0$,
(2) $\models (X,\tau)\in semi\text{-}T_2 \to (X,\tau)\in semi\text{-}T_1$.

Proof. The proof of (1) and (2) are obtained from Lemma 3.2(2) and (1), respectively. □

Corollary 3.1.

$$\models (X,\tau)\in semi\text{-}T_2 \to (X,\tau)\in semi\text{-}T_0.$$

Theorem 3.4.

$$\models (X,\tau)\in semi\text{-}T_0 \leftrightarrow (\forall x\,\forall y\,(x\in X\wedge y\in X\wedge x\neq y\to(\neg(x\in semi\text{-}cl(\{y\})))$$
$$\vee\neg(y\in semi\text{-}cl(\{x\})))).$$

Proof.

$$\begin{aligned}[(X,\tau)\in semi\text{-}T_0] &= \inf_{x\neq y}\max\left(\sup_{y\notin A} SN_x(A), \sup_{x\notin A} SN_y(A)\right)\\
&= \inf_{x\neq y}\max(SN_x(X\sim\{y\}), SN_y(X\sim\{x\}))\\
&= \inf_{x\neq y}\max(1-semi\text{-}cl(\{y\})(x), 1-semi\text{-}cl(\{x\})(y))\\
&= \inf_{x\neq y}(\neg(semi\text{-}cl(\{y\})(x)\vee\neg(semi\text{-}cl(\{x\})(y))\\
&= [\forall x\,\forall y\,(x\in X\wedge y\in X\wedge x\neq y\to(\neg(x\in semi\text{-}cl(\{y\})))\\
&\quad \vee\neg(y\in semi\text{-}cl(\{x\}))))].\end{aligned}$$ □

Theorem 3.5. *For any fuzzifying topological space* (X,τ),

$$\models (X,\tau)\in semi\text{-}T_1 \leftrightarrow \forall x\,(\{x\}\in SF).$$

Proof. For any $x_1,x_2,x_1\neq x_2$,

$$[\forall x\,(\{x\}\in SF)] = \inf_{x\in X} SF(\{x\}) = \inf_{x\in X} S\tau(X\sim\{x\})$$

$$= \inf_{x\in X} \inf_{y\in X\sim\{x\}} SN_y(X\sim\{x\}) \leqslant \inf_{y\in X\sim\{x_2\}} SN_y(X\sim\{x_2\})$$
$$\leqslant SN_{x_1}(X\sim\{x_2\}) = \sup_{x_2\notin A} SN_{x_1}(A).$$

Similarly, we have, $[\forall x(\{x\}\in SF)] \leqslant \sup_{x_1\notin B} SN_{x_2}(B)$. Then,

$$[\forall x(\{x\}\in SF)] \leqslant \inf_{x_1\neq x_2} \min\left(\sup_{x_2\notin A} SN_{x_1}(A), \sup_{x_1\notin B} SN_{x_2}(B)\right)$$
$$= \inf_{x_1\neq x_2} \sup_{x_1\notin B, x_2\notin A} \min(SN_{x_1}(A), SN_{x_2}(B)) = [(X,\tau)\in semi\text{-}T_1].$$

On the other hand,

$$[(X,\tau)\in semi\text{-}T_1] = \inf_{x_1\neq x_2} \min\left(\sup_{x_2\notin A} SN_{x_1}(A), \sup_{x_1\notin B} SN_{x_2}(B)\right)$$
$$= \inf_{x_1\neq x_2} \min(SN_{x_1}(X\sim\{x_2\}), SN_{x_2}(X\sim\{x_1\}))$$
$$\leqslant \inf_{x_1\neq x_2} SN_{x_1}(X\sim\{x_2\}) = \inf_{x_2\in X} \inf_{x_1\in X\sim\{x_2\}} SN_{x_1}(X\sim\{x_2\})$$
$$= \inf_{x_2\in X} S\tau(X\sim\{x_2\}) = \inf_{x\in X} S\tau(X\sim\{x\})$$
$$= [\forall x(\{x\}\in SF)].$$

Thus, $[(X,\tau)\in semi\text{-}T_1] = [\forall x(\{x\}\in SF)]$. □

Definition 3.2. The semi-local base $S\beta_x$ of x is a function from $P(X)$ into I such that the following conditions are satisfied:

(1) $\models S\beta_x \subseteq SN_x$,
(2) $\models A\in SN_x \to \exists B(B\in S\beta_x \wedge x\in B\subseteq A)$.

Lemma 3.2.

$$\models A\in SN_x \leftrightarrow \exists B(B\in S\beta_x \wedge x\in B\subseteq A).$$

Proof. From condition (1) in Definition 3.2 and Theorem 1.2(2) we have $SN_x(A)\geqslant SN_x(B)\geqslant S\beta_x(B)$ for each $B\in P(X)$ such that $x\in B\subseteq A$. So, $SN_x(A)\geqslant \sup_{x\in B\subseteq A} S\beta_x(B)$. From condition (2) in Definition 3.2, $SN_x(A)\leqslant \sup_{x\in B\subseteq A} S\beta_x(B)$. Hence, $SN_x(A) = \sup_{x\in B\subseteq A} S\beta_x(B)$. □

Theorem 3.6. *If $S\beta_x$ is a semi-local basis of x, then*

$$\models (X,\tau)\in semi\text{-}T_2 \leftrightarrow \forall x\forall y(x\in X \wedge y\in X \wedge x\neq y \to \exists B(B\in S\beta_x \wedge y\in \neg(semi\text{-}cl(B)))).$$

Proof.

$$[\forall x\forall y(x\in X \wedge y\in X \wedge x\neq y \to \exists B(B\in S\beta_x \wedge y\in \neg(semi\text{-}cl(B))))]$$
$$= \inf_{x\neq y} \sup_{B\in P(X)} \min(S\beta_x(B), SN_y(X\sim B))$$

$$= \inf_{x\neq y} \sup_{B\in P(X)} \sup_{y\in C\subseteq X\sim B} \min(S\beta_x(B), S\beta_y(C))$$

$$= \inf_{x\neq y} \sup_{B\cap C=\emptyset} \sup_{x\in D\subseteq B,\, y\in E\subseteq C} \min(S\beta_x(D), S\beta_y(E))$$

$$= \inf_{x\neq y} \sup_{B\cap C=\emptyset} \min\left(\sup_{x\in D\subseteq B} S\beta_x(D), \sup_{y\in E\subseteq C} S\beta_y(E)\right)$$

$$= \inf_{x\neq y} \sup_{B\cap C=\emptyset} \min(SN_x(B), SN_y(C)) = [(X,\tau)\in semi\text{-}T_2]. \qquad \square$$

Theorem 3.7.

$$\models (X,\tau)\in semi\text{-}R_1 \to (X,\tau)\in semi\text{-}R_0.$$

Proof. From Lemma 3.2(1) the proof is immediate. □

Theorem 3.8. (1) $\models (X,\tau)\in semi\text{-}T_1 \to (X,\tau)\in semi\text{-}R_0$,
(2) $\models (X,\tau)\in semi\text{-}T_1 \to (X,\tau)\in semi\text{-}R_0 \wedge (x,\tau)\in semi\text{-}T_0$,
(3) *If* $semi\text{-}T_0(X,\tau)=1$, *then*

$$\models (X,\tau)\in semi\text{-}T_1 \leftrightarrow (X,\tau)\in semi\text{-}R_0 \wedge (X,\tau)\in semi\text{-}T_0.$$

Proof. (1) Applying Lemma 2.2 we have,

$$semi\text{-}T_1(X,\tau) = \inf_{x\neq y}[SH_{x,y}] \leqslant \inf_{x\neq y}[SK_{x,y}\to SH_{x,y}] = semi\text{-}R_0(X,\tau).$$

(2) It is obtained from (1) and from Theorem 3.3(1).
(3) Since $semi\text{-}T_0(X,\tau)=1$, for every $x,y\in X$ such that $x\neq y$ we have $[SK_{x,y}]=1$. Now,

$$[(X,\tau)\in semi\text{-}R_0 \wedge (X,\tau)\in semi\text{-}T_0]$$

$$= [(X,\tau)\in semi\text{-}R_0] = \inf_{x\neq y}\min(1, 1-[SK_{x,y}]+[SH_{x,y}])$$

$$= \inf_{x\neq y}[SH_{x,y}] = semi\text{-}T_1(X,\tau). \qquad \square$$

Theorem 3.9. (1) $\models (X,\tau)\in semi\text{-}R_0 \mathbin{\dot\wedge} (X,\tau)\in semi\text{-}T_0 \to (X,\tau)\in semi\text{-}T_1$;
(2) *If* $semi\text{-}T_0(X,\tau)=1$, *then*

$$\models (X,\tau)\in semi\text{-}R_0 \mathbin{\dot\wedge} (X,\tau)\in semi\text{-}T_0 \leftrightarrow (X,\tau)\in semi\text{-}T_1.$$

Proof. (1)

$$[(X,\tau)\in semi\text{-}R_0 \mathbin{\dot\wedge} (X,\tau)\in semi\text{-}T_0]$$

$$= \max(0, semi\text{-}R_0(X,\tau) + semi\text{-}T_0(X,\tau) - 1)$$

$$= \max\left(0, \inf_{x\neq y}\min(1, 1-[SK_{x,y}]+[SH_{x,y}]) + \inf_{x\neq y}[SK_{x,y}] - 1\right)$$

$$\leqslant \max\left(0, \inf_{x\neq y}(\min(1, 1-[SK_{x,y}]+[SH_{x,y}])+[SK_{x,y}]-1)\right)$$

$$= \inf_{x\neq y}[SH_{x,y}] = semi\text{-}T_1(X,\tau).$$

(2)

$$[(X,\tau)] \in semi\text{-}R_0 \mathbin{\&} [(X,\tau)\in semi\text{-}T_0]$$

$$= [(X,\tau)\in semi\text{-}R_0] = \inf_{x\neq y}\min(1, 1-[SK_{x,y}]+[SH_{x,y}])$$

$$= \inf_{x\neq y}[SH_{x,y}] = semi\text{-}T_1(X,\tau),$$

because $semi\text{-}T_0(X,\tau)=1$, we have for each $x, y\in X$ such that $x\neq y$, $[SK_{x,y}]=1$. □

Theorem 3.10. (1) $\models (X,\tau)\in semi\text{-}T_0 \to ((X,\tau)\in semi\text{-}R_0 \to (X,\tau)\in semi\text{-}T_1)$,
(2) $\models (X,\tau)\in semi\text{-}R_0 \to ((X,\tau)\in semi\text{-}T_0 \to (X,\tau)\in semi\text{-}T_1)$.

Proof. From Theorems 3.8(1) and 3.9(1), we have
(1)

$$[(X,\tau)\in semi\text{-}T_0 \to ((X,\tau)\in semi\text{-}R_0 \to (X,\tau)\in semi\text{-}T_1)]$$

$$= \min(1, 1-[(X,\tau)\in semi\text{-}T_0] + \min(1, 1-[(X,\tau)\in semi\text{-}R_0]+[(X,\tau)\in semi\text{-}T_1]))$$

$$= \min(1, 1-[(X,\tau)\in semi\text{-}T_0] + 1-[(X,\tau)\in semi\text{-}R_0]+[(X,\tau)\in semi\text{-}T_1])$$

$$= \min(1, 1-([(X,\tau)\in semi\text{-}T_0] + [(X,\tau)\in semi\text{-}R_0]-1)+[(X,\tau)\in semi\text{-}T_1])$$

$$= 1.$$

(2)

$$[(X,\tau)\in semi\text{-}R_0 \to ((X,\tau)\in semi\text{-}T_0 \to (X,\tau)\in semi\text{-}T_1)]$$

$$= \min(1, 1-([(X,\tau)\in semi\text{-}T_0] + [(X,\tau)\in semi\text{-}R_0]-1)+[(X,\tau)\in semi\text{-}T_1])$$

$$= 1. \quad \square$$

Theorem 3.11. (1) $\models (X,\tau)\in semi\text{-}T_2 \to (X,\tau)\in semi\text{-}R_1$,
(2) $\models (X,\tau)\in semi\text{-}T_2 \to (X,\tau)\in semi\text{-}R_1 \wedge (X,\tau)\in semi\text{-}T_0$;
(3) *If* $semi\text{-}T_0(X,\tau)=1$, *then*

$$\models (X,\tau)\in semi\text{-}T_2 \leftrightarrow (X,\tau)\in semi\text{-}R_1 \wedge (X,\tau)\in semi\text{-}T_0.$$

Proof. (1) Applying Lemma 2.2 we have

$$semi\text{-}T_2(X,\tau) = \inf_{x\neq y}[SM_{x,y}] \leqslant \inf_{x\neq y}[SK_{x,y}\to SM_{x,y}] = semi\text{-}R_1(X,\tau).$$

(2) It is obtained from (1) and Corollary 3.1.
(3) Since $semi\text{-}T_0(X,\tau)=1$, then for each $x, y\in X$ such that $x\neq y$ we have $[SK_{x,y}]=1$.

Now,

$$[(X,\tau)\in semi\text{-}R_1 \wedge (X,\tau)\in semi\text{-}T_0]$$

$$=[(X,\tau)\in semi\text{-}R_1]=\inf_{x\neq y}\min(1,1-[SK_{x,y}]+[SM_{x,y}])$$

$$=\inf_{x\neq y}[SM_{x,y}]=semi\text{-}T_2(X,\tau). \quad \square$$

Theorem 3.12. (1) $\models (X,\tau)\in semi\text{-}R_1 \mathbin{\dot\wedge} (X,\tau)\in semi\text{-}T_0 \to (X,\tau)\in semi\text{-}T_2$,
(2) *If* $semi\text{-}T_0(X,\tau)=1$, *then*

$$\models (X,\tau)\in semi\text{-}R_1 \mathbin{\dot\wedge} (X,\tau)\in semi\text{-}T_0 \leftrightarrow (X,\tau)\in semi\text{-}T_2.$$

Proof. (1)

$$[(X,\tau)\in semi\text{-}R_1 \mathbin{\dot\wedge} (X,\tau)\in semi\text{-}T_0]$$

$$=\max(0,\ semi\text{-}R_1(X,\tau)+semi\text{-}T_0(X,\tau)-1)$$

$$=\max\left(0,\ \inf_{x\neq y}\min(1,1-[SK_{x,y}]+[SM_{x,y}])+\inf_{x\neq y}[SK_{x,y}]-1\right)$$

$$\leqslant\max\left(0,\ \inf_{x\neq y}(\min(1,1-[SK_{x,y}]+[SM_{x,y}])+[SK_{x,y}]-1)\right)$$

$$=\inf_{x\neq y}[SM_{x,y}]=semi\text{-}T_2(X,\tau).$$

(2)

$$[(X,\tau)\in semi\text{-}R_1 \mathbin{\dot\wedge} (X,\tau)\in semi\text{-}T_0]$$

$$=[(X,\tau)\in semi\text{-}R_1]=\inf_{x\neq y}\min(1,1-[SK_{x,y}]+[SM_{x,y}])$$

$$=\inf_{x\neq y}[SM_{x,y}]=semi\text{-}T_2(X,\tau),$$

since $semi\text{-}T_0(X,\tau)=1$, then for each $x,y\in X$ such that $x\neq y$, $[SK_{x,y}]=1$. $\square$

Theorem 3.13. (1) $\models (X,\tau)\in semi\text{-}T_0 \to ((X,\tau)\in semi\text{-}R_1 \to (X,\tau)\in semi\text{-}T_2)$,
(2) $\models (X,\tau)\in semi\text{-}R_1 \to ((X,\tau)\in semi\text{-}T_0 \to (X,\tau)\in semi\text{-}T_2)$.

Proof. (1) From Theorems 3.11(1), 3.12(1) we have

$$[(X,\tau)\in semi\text{-}T_0 \to ((X,\tau)\in semi\text{-}R_1 \to (X,\tau)\in semi\text{-}T_2)]$$

$$=\min(1,1-[(X,\tau)\in semi\text{-}T_0]+\min(1,1-[(X,\tau)\in semi\text{-}R_1]+[(X,\tau)\in semi\text{-}T_2]))$$

$$=\min(1,1-[(X,\tau)\in semi\text{-}T_0]+1-[(X,\tau)\in semi\text{-}R_1]+[(X,\tau)\in semi\text{-}T_2])$$

$$=\min(1,1-([(X,\tau)\in semi\text{-}T_0]+[(X,\tau)\in semi\text{-}R_1]-1)+[(X,\tau)\in semi\text{-}T_2])=1.$$

(2)

$$\begin{aligned}
&[(X,\tau)\in semi\text{-}R_1 \rightarrow ((X,\tau)\in semi\text{-}T_0 \rightarrow (X,\tau)\in semi\text{-}T_2)]\\
&\quad = \min(1, 1-[(X,\tau)\in semi\text{-}R_1] + \min(1, 1-[(X,\tau)\in semi - T_0] + [(X,\tau)\in semi\text{-}T_2]))\\
&\quad = \min(1, 1-[(X,\tau)\in semi\text{-}R_1] + 1-[(X,\tau)\in semi\text{-}T_0] + [(X,\tau)\in semi\text{-}T_2])\\
&\quad = \min(1, 1-([(X,\tau)\in semi\text{-}R_1] + [(X,\tau)\in semi\text{-}T_0] - 1) + [(X,\tau)\in semi\text{-}T_2]) = 1. \qquad \square
\end{aligned}$$

Theorem 3.14. *If* $semi\text{-}T_0(X,\tau)=1$, *then*

(1) $\models ((X,\tau)\in semi\text{-}T_0 \rightarrow ((X,\tau)\in semi\text{-}R_0 \rightarrow (X,\tau)\in semi\text{-}T_1))$
$\wedge ((X,\tau)\in semi\text{-}T_1 \rightarrow \neg((X,\tau)\in semi\text{-}T_0 \rightarrow \neg((X,\tau)\in semi\text{-}R_0)))$,

(2) $\models ((X,\tau)\in semi\text{-}R_0 \rightarrow ((X,\tau)\in semi\text{-}T_0 \rightarrow (X,\tau)\in semi\text{-}T_1))$
$\wedge ((X,\tau)\in semi\text{-}T_1 \rightarrow \neg((X,\tau)\in semi\text{-}T_0 \rightarrow \neg((X,\tau)\in semi\text{-}R_0)))$,

(3) $\models ((X,\tau)\in semi\text{-}T_0 \rightarrow ((X,\tau)\in semi\text{-}R_0 \rightarrow (X,\tau)\in semi\text{-}T_1))$
$\wedge ((X,\tau)\in semi\text{-}T_1 \rightarrow \neg((X,\tau)\in semi\text{-}R_0 \rightarrow \neg((X,\tau)\in semi\text{-}T_0)))$,

(4) $\models ((X,\tau)\in semi\text{-}R_0 \rightarrow ((X,\tau)\in semi\text{-}T_0 \rightarrow (X,\tau)\in semi\text{-}T_1))$
$\wedge ((X,\tau)\in semi\text{-}T_1 \rightarrow \neg((X,\tau)\in semi\text{-}R_0 \rightarrow \neg((X,\tau)\in semi\text{-}T_0)))$.

Proof. For simplicity we put, $semi\text{-}T_0(X,\tau)=\alpha$, $semi\text{-}R_0(X,\tau)=\beta$ and $semi\text{-}T_1(X,\tau)=\gamma$. Now, applying Theorem 3.9(2), the proof is obtained with some relations in fuzzy logic as follows:
(1)

$$\begin{aligned}
1 &= (\alpha \mathbin{\dot\wedge} \beta \leftrightarrow \gamma) = (\alpha \mathbin{\dot\wedge} \beta \rightarrow \gamma) \wedge (\gamma \rightarrow \alpha \mathbin{\dot\wedge} \beta)\\
&= \neg(\alpha \mathbin{\dot\wedge} \beta \mathbin{\dot\wedge} \neg\gamma) \wedge \neg(\gamma \wedge \neg(\alpha \mathbin{\dot\wedge} \beta)) = \neg(\alpha \mathbin{\dot\wedge} \neg(\neg(\beta \mathbin{\dot\wedge} \neg\gamma))) \wedge \neg(\gamma \mathbin{\dot\wedge} (\alpha \rightarrow \neg\beta))\\
&= (\alpha \rightarrow \neg(\beta \mathbin{\dot\wedge} \neg\gamma)) \wedge (\gamma \rightarrow \neg(\alpha \rightarrow \neg\beta) = (\alpha \rightarrow (\beta \rightarrow \gamma) \wedge (\gamma \rightarrow \neg(\alpha \rightarrow \neg\beta)),
\end{aligned}$$

since $\dot\wedge$ is commutative one can have the proof of statements (2)–(4) in a similar way as (1). $\square$

By a similar procedure to Theorem 3.13 one can have the following theorem.

Theorem 3.15.

(1) $\models ((X,\tau)\in semi\text{-}T_0 \rightarrow ((X,\tau)\in semi\text{-}R_1 \rightarrow (X,\tau)\in semi\text{-}T_2))$
$\wedge ((X,\tau)\in semi\text{-}T_2 \rightarrow \neg((X,\tau)\in semi\text{-}T_0 \rightarrow \neg((X,\tau)\in semi\text{-}R_1)))$,

(2) $\models ((X,\tau)\in semi\text{-}R_1 \rightarrow ((X,\tau)\in semi\text{-}T_0 \rightarrow (X,\tau)\in semi\text{-}T_2))$
$\wedge((X,\tau)\in semi\text{-}T_2 \rightarrow \neg((X,\tau)\in semi\text{-}T_0 \rightarrow \neg((X,\tau)\in semi\text{-}R_1)))$,

(3) $\models ((X,\tau)\in semi\text{-}T_0 \rightarrow ((X,\tau)\in semi\text{-}R_1 \rightarrow (X,\tau)\in semi\text{-}T_2))$
$\wedge((X,\tau)\in semi\text{-}T_2 \rightarrow \neg((X,\tau)\in semi\text{-}R_1 \rightarrow \neg((X,\tau)\in semi\text{-}T_0)))$,

(4) $\models ((X,\tau)\in semi\text{-}R_1 \rightarrow ((X,\tau)\in semi\text{-}T_0 \rightarrow (X,\tau)\in semi\text{-}T_2))$
$\wedge((X,\tau)\in semi\text{-}T_2 \rightarrow \neg((X,\tau)\in semi\text{-}R_1 \rightarrow \neg((X,\tau)\in semi\text{-}T_0)))$.

Lemma 3.4. (1) *If* $D \subseteq B$, *then* $\sup_{A\cap B=\emptyset} SN_x(A) = \sup_{A\cap B=\emptyset, D\subseteq B} SN_x(A)$;
(2) $\sup_{A\cap B=\emptyset} \inf_{y\in D} SN_y(X\sim A) = \sup_{A\cap B=\emptyset, D\subseteq B} S\tau(B)$.

Proof. (1) Since $D\subseteq B$,

$$\sup_{A\cap B=\emptyset} SN_x(A) = \sup_{A\cap B=\emptyset} SN_x(A) \wedge [D\subseteq B] = \sup_{A\cap B=\emptyset, D\subseteq B} SN_x(A)$$

(2) Let $y\in D$ and $A\cap B=\emptyset$.
Then,

$$\begin{aligned}\sup_{A\cap B=\emptyset, D\subseteq B} S\tau(B) &= \sup_{A\cap B=\emptyset, D\subseteq B} S\tau(B)\wedge[y\in D]\\ &= \sup_{y\in D\subseteq B\subseteq X\sim A} S\tau(B) = \sup_{y\in B\subseteq X\sim A} S\tau(B)\\ &= SN_y(X\sim A) = \inf_{y\in D} SN_y(X\sim A)\\ &= \sup_{A\cap B=\emptyset}\inf_{y\in D} SN_y(X\sim A). \qquad \square\end{aligned}$$

Definition 3.4.

$$semi\text{-}T_3^{(1)}(X,\tau) := \forall x\,\forall D(x\in X\wedge D\in F\wedge x\notin D\to \exists A(A\in SN_x\wedge(D\subseteq X\sim semi\text{-}cl(A)))).$$

Theorem 3.16.

$$\models (X,\tau)\in semi\text{-}T_3 \leftrightarrow (X,\tau)\in semi\text{-}T_3^{(1)}.$$

Proof. Now,

$$\begin{aligned}semi\text{-}T_3^{(1)}(X,\tau) &= \inf_{x\notin D}\min\left(1, 1-\tau(X\sim D)+\sup_{A\in P(X)}\min\left(SN_x(A), \inf_{y\in D}(1-semi\text{-}cl(A)(y))\right)\right),\\ &= \inf_{x\notin D}\min\left(1, 1-\tau(X\sim D)+\sup_{A\in P(X)}\min\left(SN_x(A), \inf_{y\in D} SN_y(X\sim A)\right)\right),\end{aligned}$$

and

$$semi\text{-}T_3(X,\tau) = \inf_{x\notin D}\min\left(1, 1-\tau(X\sim D)+\sup_{A\cap B=\emptyset, D\subseteq B}\min(SN_x(A), S\tau(B))\right).$$

So, the result holds if we prove that

$$\sup_{A\in P(X)}\min\left(SN_x(A), \inf_{y\in D} SN_y(X\sim A)\right) = \sup_{A\cap B=\emptyset, D\subseteq B}\min(SN_x(A), S\tau(B)). \qquad (*)$$

It is clear that, on the left-hand side of $(*)$ when $A\cap D\neq\emptyset$ then there exists $y\in X$ such that $y\in D$ and $y\notin X\sim A$. So, $\inf_{y\in D} SN_y(X\sim A)=0$ and thus $(*)$ becomes

$$\sup_{A\in P(X), A\cap B=\emptyset}\min\left(SN_x(A), \inf_{y\in D} SN_y(X\sim A)\right) = \sup_{A\cap B=\emptyset, D\subseteq B}\min(SN_x(A), S\tau(B)),$$

which is obtained from Lemma 3.4. $\square$

Definition 3.5.

$$semi\text{-}T_3^{(2)}(X,\tau) := \forall x\, \forall B(x \in B \wedge B \in \tau \to \exists A(A \in SN_x \wedge semi\text{-}cl(A) \subseteq B)).$$

Theorem 3.17.

$$\models (X,\tau) \in semi\text{-}T_3 \leftrightarrow (X,\tau) \in semi\text{-}T_3^{(2)}.$$

Proof. From Theorem 3.16, we have

$$semi\text{-}T_3(X,\tau) = \inf_{x \notin D} \min\left(1, 1 - \tau(X \sim D) + \sup_{A \in P(X)} \min\left(SN_x(A), \inf_{y \in D} SN_y(X \sim A)\right)\right).$$

Now, if we put $B = X{\sim}D$, then

$$\begin{aligned} semi\text{-}T_3^{(2)}(X,\tau) &= \inf_{x \in B} \min\left(1, 1 - \tau(B) + \sup_{A \in P(X)} \min\left(SN_x(A), \inf_{y \in X \sim B} SN_y(X \sim A)\right)\right) \\ &= \inf_{x \notin D} \min\left(1, 1 - \tau(X \sim D) + \sup_{A \in P(X)} \min\left(SN_x(A), \inf_{y \in D} SN_y(X \sim A)\right)\right) \\ &= semi\text{-}T_3(X,\tau). \qquad \square \end{aligned}$$

Definition 3.6. Let φ be a subbase of τ then,

$$semi\text{-}T_3^{(3)}(X,\tau) := \forall x\, \forall D(x \in D \wedge D \in \varphi \to \exists B(B \in SN_x \wedge semi\text{-}cl(B) \subseteq D).$$

Theorem 3.18.

$$\models (X,\tau) \in semi\text{-}T_3 \leftrightarrow (X,\tau) \in semi\text{-}T_3^{(3)}.$$

Proof. Since $[\varphi \subseteq \tau] = 1$, and with regard to Theorems 3.16 and 3.17 $semi\text{-}T_3^{(3)}(X,\tau) \geqslant semi\text{-}T_3^{(2)}(X,\tau) = semi\text{-}T_3(X,\tau)$. So, it remains to prove that $semi\text{-}T_3^{(3)}(X,\tau) \leqslant semi\text{-}T_3^{(2)}(X,\tau)$ and this is obtained if we prove for any $x \in A$,

$$\min\left(1, 1 - \tau(A) + \sup_{B \in P(X)} \min\left(SN_x(B), \inf_{y \in X \sim A} SN_y(X{\sim}B)\right)\right) \geqslant semi\text{-}T_3^{(3)}(X,\tau).$$

Set $semi\text{-}T_3^{(3)}(X,\tau) = \delta$. Then, for any $x \in X$ and any $D_{\lambda_i} \in P(X)$, $\lambda_i \in I_\lambda$ (I_λ denotes a finite index set), $\lambda \in \Lambda$, $\bigcup_{\lambda \in \Lambda} \bigcap_{\lambda_i \in I_\lambda} D_{\lambda_i} = A$ we have,

$$1 - \varphi(D_{\lambda_i}) + \sup_{B \in P(X)} \min\left(SN_x(B), \inf_{y \in X \sim D_{\lambda_i}} SN_y(X \sim B)\right) \geqslant \delta > \delta - \varepsilon,$$

where ε is any positive number. Thus,

$$\sup_{B \in P(X)} \min\left(SN_x(B), \inf_{y \in X \sim D_{\lambda_i}} SN_y(X \sim B)\right) > \varphi(D_{\lambda_i}) - 1 + \delta - \varepsilon.$$

Set $\gamma_{\lambda_i}=\{B\colon B\subseteq D_{\lambda_i}\}$. Then,

$$\begin{aligned}
&\inf_{\lambda_i\in I_\lambda}\sup_{B\in P(X)}\min\left(SN_x(B),\inf_{y\in X\sim D_{\lambda_i}}SN_y(X\sim B)\right)\\
&\quad=\sup_{f\in\Pi\{\gamma_{\lambda_i}\colon\,\lambda_i\in I_\lambda\}}\inf_{\lambda_i\in I_\lambda}\min\left(SN_x(f(\lambda_i)),\inf_{y\in X\sim D_{\lambda_i}}SN_y(X\sim f(\lambda_i))\right)\\
&\quad=\sup_{f\in\Pi\{\gamma_{\lambda_i}\colon\,\lambda_i\in I_\lambda\}}\min\left(\inf_{\lambda_i\in I_\lambda}SN_x(f(\lambda_i)),\inf_{\lambda_i\in I_\lambda}\inf_{y\in X\sim D_{\lambda_i}}SN_y(X\sim f(\lambda_i))\right)\\
&\quad=\sup_{f\in\Pi\{\gamma_{\lambda_i}\colon\,\lambda_i\in I_\lambda\}}\min\left(\inf_{\lambda_i\in I_\lambda}SN_x(f(\lambda_i)),\inf_{y\in\bigcup_{\lambda_i\in I_\lambda}X\sim D_{\lambda_i}}SN_y(X\sim f(\lambda_i))\right)\\
&\quad=\sup_{B\in P(X)}\min\left(\inf_{\lambda_i\in I_\lambda}SN_x(B),\inf_{y\in\bigcup_{\lambda_i\in I_\lambda}X\sim D_{\lambda_i}}SN_y(X\sim B)\right)\\
&\quad=\sup_{B\in P(X)}\min\left(SN_x(B),\inf_{y\in\bigcup_{\lambda_i\in I_\lambda}X\sim D_{\lambda_i}}SN_y(X\sim B)\right),
\end{aligned}$$

where $B=f(\lambda_i)$.

Similarly, we can prove

$$\begin{aligned}
&\inf_{\lambda\in\Lambda}\sup_{B\in P(X)}\min\left(SN_x(B),\inf_{y\in\bigcup_{\lambda_i\in I_\lambda}X\sim D_{\lambda_i}}SN_y(X\sim B)\right)\\
&\quad=\sup_{B\in P(X)}\min\left(SN_x(B),\inf_{y\in\bigcup_{\lambda\in\Lambda}\bigcup_{\lambda_i\in I_\lambda}X\sim D_{\lambda_i}}SN_y(X\sim B)\right)\\
&\quad\leqslant\sup_{B\in P(X)}\min\left(SN_x(B),\inf_{y\in\bigcap_{\lambda\in\Lambda}\bigcup_{\lambda_i\in I_\lambda}X\sim D_{\lambda_i}}SN_y(X\sim B)\right)\\
&\quad\leqslant\sup_{B\in P(X)}\min\left(SN_x(B),\inf_{y\in X\sim A}SN_y(X\sim B)\right),
\end{aligned}$$

so we have

$$\begin{aligned}
\sup_{B\in P(X)}\min\left(SN_x(B),\inf_{y\in X\sim A}SN_y(X\sim B)\right)&\geqslant\inf_{\lambda\in\Lambda}\inf_{\lambda_i\in I_\lambda}\sup_{B\in P(X)}\min\left(SN_x(B),\inf_{y\in X\sim D_{\lambda_i}}SN_y(X\sim B)\right)\\
&\geqslant\inf_{\lambda\in\Lambda}\inf_{\lambda_i\in I_\lambda}\varphi(D_{\lambda_i})-1+\delta-\varepsilon.
\end{aligned}$$

For any I_λ and Λ that satisfy $\bigcup_{\lambda\in\Lambda}\bigcap_{\lambda_i\in I_\lambda}D_{\lambda_i}=A$ the above inequality is true.

So,

$$\begin{aligned}
\sup_{B\in P(X)}\min\left(SN_x(B),\inf_{y\in X\sim A}SN_y(X\sim B)\right)&\geqslant\sup_{\bigcup_{\lambda\in\Lambda}D_\lambda\neq A}\inf_{\lambda\in\Lambda}\sup_{\bigcap_{\lambda_i\in I_\lambda}D_{\lambda_i}=D_\lambda}\inf_{\lambda_i\in I_\lambda}\varphi(D_{\lambda_i})-1+\delta-\varepsilon\\
&=\tau(A)-1+\delta-\varepsilon.
\end{aligned}$$

i.e.,

$$\min\left(1, 1-\tau(A)+\sup_{B\in P(X)}\min\left(SN_x(B), \inf_{y\in X\sim A} SN_y(X\sim B)\right)\right)\geqslant \delta-\varepsilon.$$

Because ε is any positive number, when $\varepsilon\to 0$ we have

$$semi\text{-}T_3^{(2)}(X,\tau)\geqslant\delta = semi\text{-}T_3^{(3)}(X,\tau).$$

So,

$$\models (X,\tau)\in semi\text{-}T_3 \leftrightarrow (X,\tau)\in semi\text{-}T_3^{(3)}. \qquad \square$$

Definition 3.7. Let (X,τ) be any fuzzifying topological space and let

$$semi\text{-}T_4^{(1)}(X,\tau) := \forall A\,\forall B(A\in\tau\wedge B\in F\wedge A\cap B\equiv\emptyset\to\exists G(G\in\tau\wedge A\subseteq G \wedge B\subseteq X\sim semi\text{-}cl(G)));$$

$$semi\text{-}T_4^{(2)}(X,\tau) := \forall A\,\forall B(A\in F\wedge B\in\tau\wedge A\subseteq B\to\exists G(G\in\tau\wedge A\subseteq G\wedge semi\text{-}cl(G)\subseteq B)).$$

Theorem 3.19.

$$\models (X,\tau)\in semi\text{-}T_4 \leftrightarrow (X,\tau)\in semi\text{-}T_4^{(i)},$$

where $i = 1, 2$.

Proof. The proof is similar to that of Theorems 3.16 and 3.17. $\square$

4. Relation among separation axioms

Lemma 4.1. *For every* $\alpha,\gamma\in I$ *we have*,

$$(1\wedge(1-\alpha+\gamma))+\alpha\leqslant 1+\gamma.$$

Theorem 4.1.

$$\models (X,\tau)\in semi\text{-}T_3 \mathbin{\dot\wedge} (X,\tau)\in T_1\to(X,\tau)\in semi\text{-}T_2.$$

Proof. From Theorem 2.2 [7] we have, $T_1(X,\tau) = \inf_{y\in X}\tau(X\sim\{y\})$ and applying Lemma 4.1 we have,

$$\begin{aligned}
&semi\text{-}T_3(X,\tau)+T_1(X,\tau)\\
&\quad=\inf_{x\notin D}\min\left(1, 1-\tau(X\sim D)+\sup_{A\cap B=\emptyset, D\subseteq B}\min(SN_x(A), S\tau(B))\right)+\inf_{y\in X}\tau(X\sim\{y\})\\
&\quad\leqslant\inf_{x\in X, x\neq y}\inf_{y\in X}\min\left(1, 1-\tau(X\sim\{y\})+\sup_{A\cap B=\emptyset}\min(SN_x(A), SN_y(B))\right)+\inf_{y\in X}\tau(X\sim\{y\})\\
&\quad=\inf_{x\in X, x\neq y}\left(\inf_{y\in X}\min\left(1, 1-\tau(X\sim\{y\})+\sup_{A\cap B=\emptyset}\min(SN_x(A), SN_y(B))\right)+\inf_{y\in X}\tau(X\sim\{y\})\right)
\end{aligned}$$

$$\leqslant \inf_{x\in X, x\neq y}\inf_{y\in X}\left(\min\left(1, 1-\tau(X\sim\{y\})+\sup_{A\cap B=\emptyset}\min(SN_x(A), SN_y(B))\right)+\tau(X\sim\{y\})\right)$$

$$\leqslant \inf_{x\neq y}\left(1+\sup_{A\cap B=\emptyset}\min(SN_x(A), SN_y(B))\right) = 1+\inf_{x\neq y}\sup_{A\cap B=\emptyset}\min(SN_x(A), SN_y(B))$$

$$= 1 + semi\text{-}T_2(X,\tau),$$

namely, $semi\text{-}T_2(X,\tau)\geqslant semi\text{-}T_3(X,\tau)+T_1(X,\tau)-1$. Thus, $semi\text{-}T_2(X,\tau)\geqslant \max(0,\ semi\text{-}T_3(X,\tau)+T_1(X,\tau)-1)$. □

Theorem 4.2.

$$\models (X,\tau)\in semi\text{-}T_4 \mathbin{\&} (X,\tau)\in T_1 \to (X,\tau)\in semi\text{-}T_3.$$

Proof. It is equivalent to prove that $semi\text{-}T_3(X,\tau)\geqslant semi\text{-}T_4(X,\tau)+T_1(X,\tau)-1$. In fact,

$$semi\text{-}T_4(X,\tau)+T_1(X,\tau)$$

$$= \inf_{E\cap D=\emptyset}\min\left(1, 1-\min(\tau(X\sim E),\tau(X\sim D)) + \sup_{A\cap B=\emptyset, E\subseteq A, D\subseteq B}\min(S\tau(A), S\tau(B))\right)+\inf_{z\in X}\tau(X\sim\{z\})$$

$$\leqslant \inf_{x\notin D}\min\left(1, 1-\min(\tau(X\sim\{x\}),\tau(X\sim D)) + \sup_{A\cap B=\emptyset, D\subseteq B}\min(SN_x(A), S\tau(B))\right)+\inf_{z\in X}\tau(X\sim\{z\})$$

$$\leqslant \inf_{x\notin D}\min\left(1, \max\left(1-\tau(X\sim D)+\sup_{A\cap B=\emptyset, D\subseteq B}\min(SN_x(A), S\tau(B)), 1-\tau(X\sim\{x\}) + \sup_{A\cap B=\emptyset, D\subseteq B}\min(SN_x(A), S\tau(B))\right)\right)+\inf_{z\in X}\tau(X\sim\{z\})$$

$$= \inf_{x\notin D}\max\left(\min\left(1, 1-\tau(X\sim D)+\sup_{A\cap B=\emptyset, D\subseteq B}\min(SN_x(A), S\tau(B)),\ \min(1, 1-\tau(X\sim\{x\}) + \sup_{A\cap B=\emptyset, D\subseteq B}\min(SN_x(A), S\tau(B))\right)\right)+\inf_{z\in X}\tau(X\sim\{z\})$$

$$\leqslant \inf_{x\notin D}\max\left(\min\left(1, 1-\tau(X\sim D)+\sup_{A\cap B=\emptyset, D\subseteq B}\min(SN_x(A), S\tau(B))+\tau(X\sim\{x\}),\right.\right.$$
$$\left.\left.\min\left(1, 1-\tau(X\sim\{x\}) + \sup_{A\cap B=\emptyset, D\subseteq B}\min(SN_x(A), S\tau(B))\right)\right)+\tau(X\sim\{x\})\right)$$

$$\leqslant \inf_{x \notin D} \max\left(\min\left(1, 1 - \tau(X \sim D) + \sup_{A \cap B = \emptyset, D \subseteq B} \min(SN_x(A), S\tau(B)) + \tau(X \sim \{x\}), \right.\right.$$

$$\left.\left. 1 + \sup_{A \cap B = \emptyset, D \subseteq B} \min(SN_x(A), S\tau(B)) \right) \right)$$

$$\leqslant \inf_{x \notin D} \left(\min\left(1, 1 - \tau(X \sim D) + \sup_{A \cap B = \emptyset, D \subseteq B} \min(SN_x(A), S\tau(B)) \right) + 1 \right)$$

$$\leqslant \inf_{x \notin D} \min\left(1, 1 - \tau(X \sim D) + \sup_{A \cap B = \emptyset, D \subseteq B} \min(SN_x(A), S\tau(B)) \right) + 1$$

$$= semi\text{-}T_3(X, \tau) + 1. \qquad \square$$

5. Conclusion

The role or the meaning of each theorem in the present paper is obtained from its generalization to a corresponding theorem in the crisp setting.

For example: in the crisp setting, a topological space (X, τ) is *semi*-T_1 if and only if for each $z \in X$, $\{z\} \in F$, where F is the family of closed sets. This theorem can be rewritten as follows: the truth value of a topological space (X, τ) to be *semi*-T_1 equal the infimum of the truth values of its singletons to be closed sets, where the set of truth values is $\{0, 1\}$. Now, is this theorem still valid in fuzzifying setting, i.e., if the set of truth values is $[0, 1]$?. The answer of this question is positive and is given in Theorem 3.5 above. Another example is given in Remark 2.1 and Counterexample 2.1.

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