JOURNAL OF THE EGYPTIAN MATHEMATICAL SOCIETY
(J. Egypt. Math. Soc.) Vol.15(1)(2007) pp. 41–56.

Pre-Irresolutness and Strong Compactness Fuzzifying Topology

S. A. ABD EL-BAKI AND O. R. SAYED
Department of Mathematics, Faculty of Science, Assiut University, Assiut 71516
EGYPT, E-mail: o_r_sayed@yahoo.com

Abstract

In this paper the concepts of pre-irresolute functions and strong compactness in the framework of fuzzifying topology were characterized in terms of pre-open sets. Some properties of fuzzifying pre-irresolute functions and fuzzifying strong compactness are discussed.

Keywords and Phrases: Fuzzy logic; fuzzifying topology; pre-irresoluteness; fuzzifying compactness; strong compactness.

(2000) Mathematics Subject Classification: 54A40, 54A05.

1 Introduction

In 1991, Ying [8] used the semantic method of continuous valued logic to propose the so-called fuzzifying topology as a preliminary of the research on bifuzzy topology and elementally develop topology in the theory of fuzzy sets from completely different direction. Briefly speaking, a fuzzifying topology on a set X assigns each crisp subset of X to a certain degree of being open, other than being definitely open or not. Furthermore, Ying [11] introduced the concepts of compactness and established a generalization of Tychonoff's theorem in the framework of fuzzifying topology. The class of pre-irresolute mapping was introduced by Rielly and Vamanamurthy [7]. Park and Park [6] introduced the concept of fuzzy pre-irresolute mapping. The concept of strong compactness for topological spaces has been discussed in [3, 4]. In [5, 12] the concept of strong compactness for fuzzy topological spaces were introduced and discussed . In [1] the concepts of fuzzifying pre-open set and fuzzifying pre-continuity were introduced and studied. Also, in [2] the concept of fuzzifying pre-Hausdorff separation axiom was introduced and studied. In this paper we introduce and study the concept of pre-irresolute function between fuzzifying topological spaces. Furthermore, we introduce and study the concept of strong compactness in the framework of fuzzifying topology. We use the finite intersection property to give a characterization of the fuzzifying strong compactness.

2 Preliminaries

We present the fuzzy logical and corresponding set theoretical notations [8, 9] since we need them in this paper. For any formula φ, the symbol $[\varphi]$ means the truth value of φ, where the set of truth values is the unit interval [0, 1]. We write $\models \varphi$ if $[\varphi] = 1$ for any interpretation. By $\models^{w} \varphi$(φ is feebly valid) we mean that for any valuation it always holds that $[\varphi] > 0$, and $\varphi \models^{ws} \psi$ we mean that $[\varphi] > 0$ implies $[\psi] = 1$. The original formulae of fuzzy logical and corresponding set theoretical notations are:

(1) (a) $[\alpha] = \alpha(\alpha \in [0,1])$;
(b) $[\varphi \wedge \psi] = \min([\varphi], [\psi])$;
(c) $[\varphi \rightarrow \psi] = \min(1, 1 - [\varphi] + [\psi])$.

(2) If $\widetilde{A} \in \Im(X), [x \in \widetilde{A}] := \widetilde{A}(x)$.

(3) If X is the universe of discourse, then $[\forall x \varphi(x)] := \inf_{x \in X} [\varphi(x)]$.

In addition the following derived formulae are given,

(1) $[\neg\varphi] := [\varphi \rightarrow 0] = 1 - [\varphi]$;

(2) $[\varphi \vee \psi] := [\neg(\neg\varphi \wedge \neg\psi)] = \max([\varphi], [\psi])$;

(3) $[\varphi \leftrightarrow \psi] := [(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)]$;

(4) $[\varphi \mathbin{\dot\wedge} \psi] := [\neg(\varphi \rightarrow \neg\psi) = \max(0, [\varphi] + [\psi] - 1)$;

(5) $[\exists x \varphi(x)] := [\neg\forall x \neg\varphi(x)] := \sup_{x \in X} [\varphi(x)]$;

(6) If $\widetilde{A}, \widetilde{B} \in \Im(X)$, then
$[\widetilde{A} \subseteq \widetilde{B}] := [\forall x(x \in \widetilde{A} \rightarrow x \in \widetilde{B})] = \inf_{x \in X} \min(1, 1 - \widetilde{A}(x) + \widetilde{B}(x))$,
where $\Im(X)$ is the family of all fuzzy sets in X.

Definition 1. *[8]. Let X be a universe of discourse, $\tau \in \Im(P(X))$ satisfy the following conditions:*

(1) $\tau(X) = 1, \tau(\phi) = 1$;

(2) for any $A, B, \tau(A \cap B) \geq \tau(A) \wedge \tau(B)$;

(3) for any $\{A_\lambda : \lambda \in \Lambda\}, \tau\left(\bigcup_{\lambda \in \Lambda} A_\lambda\right) \geq \bigwedge_{\lambda \in \Lambda} \tau(A_\lambda)$.

Then τ is called a fuzzifying topology and (X, τ) is a fuzzifying topological space.

Definition 2. *[8]. The family of all fuzzifying closed sets, denoted by $\digamma \in \Im(P(X))$, is defined as follows: $A \in \digamma := X - A \in \tau$, where $X - A$ is the complement of A.*

Definition 3. *[8]. The fuzzifying neighborhood system of a point $x \in X$ is denoted by $N_x \in \Im(P(X))$ and defined as follows: $N_x(A) = \sup_{x \in B \subseteq A} \tau(B)$.*

Definition 4 (8, Lemma 5.2). . *The closure $\overline{A}$ of A is defined as $\overline{A}(x) = 1 - N_x(X - A)$. In Theorem 5.3[8], Ying proved that the closure^{-}: $P(X) \rightarrow \Im(X)$ is a fuzzifying closure operator (see Definition 5.3 [8]) because its extension*

$^{-}: \Im(X) \to \Im(X), \overline{\widetilde{A}} = \bigcup_{\alpha\in[0,1]} \alpha\overline{\widetilde{A}_\alpha}, \widetilde{A} \in \Im(X)$, *where* $\widetilde{A}_\alpha = \{x : \widetilde{A}(x) \geq \alpha\}$ *is the* α*-cut of* $\tilde{A}$ *and* $\alpha\widetilde{A}(x) = \alpha \wedge \widetilde{A}(x)$ *satisfies the following Kuratowski closure axioms:*

(1) $\models \overline{\phi} = \phi$;

(2) for any $\widetilde{A} \in \Im(X), \models \widetilde{A} \subseteq \overline{\widetilde{A}}$;

(3) for any $\widetilde{A}, \widetilde{B} \in \Im(X)$, $\models \overline{\widetilde{A} \cup \widetilde{B}} \equiv \overline{\widetilde{A}} \cup \overline{\widetilde{B}}$

(4) for any $\widetilde{A}, \widetilde{B} \in \Im(X)$, $\models \overline{\left(\overline{\widetilde{A}}\right)} \subseteq \widetilde{A}$.

Definition 5. *[9]. For any* $A \subseteq X$*, the fuzzy set of interior points of* A *is called the interior of* A*, and given as follows:* $A^\circ(x) := N_x(A)$.

From Lemma 3.1 [8] and the definitions of $N_x(A)$ *and* A° *we have* $\tau(A) = \inf_{x\in A} A^\circ(x)$.

Definition 6. *[1]. For any* $\widetilde{A} \in \Im(X), \models \left(\widetilde{A}\right)^\circ \equiv X - \overline{\left(X - \widetilde{A}\right)}$.

Lemma 1. *[1]. If* $[\widetilde{A} \subseteq \widetilde{B}] = 1$*, then (1)* $\models \overline{\widetilde{A}} \subseteq \overline{\widetilde{B}}$ *(2)* $\models \left(\widetilde{A}\right)^\circ \subseteq \left(\widetilde{B}\right)^\circ$.

Definition 7. *[1]. Let* (X, τ) *be a fuzzifying topological space.*

(1) The family of all fuzzifying pre-open sets, denoted by $\tau_P \in \Im(P(X))$*, is defined as follows:*

$A \in \tau_P := \forall x(x \in A \to x \in A^{-\circ})$*, i. e.,* $\tau_P(A) = \inf_{x\in A} A^{-\circ}(x)$

(2) The family of all fuzzifying pre-closed sets, denoted by $\digamma_P \in \Im(P(X))$*, is defined as follows:* $A \in \digamma_P := X - A \in \tau_P$.

(3) The fuzzifying pre-neighborhood system of a point $x \in X$ *is denoted by* $N_x^P \in \Im(P(X))$ *and defined as follows:* $N_x^P(A) = \sup_{x\in A} \tau_P(A)$.

(4) The fuzzifying pre-closure of a set $A \in P(X)$ *, denoted by* $Cl_P \in \Im(X)$*, is defined as follows:* $Cl_P(A)(x) = 1 - N_x^P(X - A)$.

(5) Let (X, τ) *and* (Y, σ) *be two fuzzifying topological spaces and let* $f \in Y^X$*. A unary fuzzy predicate* $C_P \in \Im(Y^X)$*, called fuzzifying pre-continuity, is given as follows:* $C_P(f) := \forall B(B \in \sigma \to f^{-1}(B) \in \tau_P)$.

Definition 8. *[2]. Let* Ω *be the class of all fuzzifying topological spaces. The unary fuzzy predicate* T_2^P *(fuzzifying pre-Hausdorff)*$\in \Im(\Omega)$ *is defined as follows:*

$T_2^P(X, \tau) := \forall x \forall y((x \in X \wedge y \in X \wedge x \neq y) \longrightarrow \exists B \exists C(B \in N_x^P \wedge C \in N_y^P \wedge B \cap C = \phi))$.

Definition 9. *[11]. Let* X *be a set. If* $\widetilde{A} \in \Im(X)$ *such that the support supp* $\widetilde{A} = \{x \in X : \widetilde{A}(x) > 0\}$ *of* A *is finite, then* $\widetilde{A}$ *is said to be finite and we write* $F(\widetilde{A})$*. A unary fuzzy predicate* $FF \in \Im(\Im(X))$*, called fuzzy finiteness, is given as* $FF(\widetilde{A}) := (\exists\widetilde{B})(F(\widetilde{B}) \wedge (\widetilde{A} \equiv \widetilde{B})) = 1 - \inf\{\alpha \in [0,1] : F(\widetilde{A}_\alpha)\} = 1 - \inf\{\alpha \in [0,1] : F(\widetilde{A}_{[\alpha]})\}$*, where* $\widetilde{A}_\alpha = \{x \in X : \widetilde{A}(x) \geq \alpha\}$ *and* $\widetilde{A}_{[\alpha]} = \{x \in X : \widetilde{A}(x) > \alpha\}$.

Definition 10. *[11]. Let X be a set.*

(1) A binary fuzzy predicate $K \in \Im(\Im(P(X)) \times P(X))$, called fuzzifying covering, is given as follows: $K(\Re, A) := \forall x(x \in A \longrightarrow \exists B(B \in \Re \wedge x \in B))$.

(2) Let (X,τ) be a fuzzifying topological space. A binary fuzzy predicate $K_\circ \in \Im(\Im(P(X)) \times P(X))$, called fuzzifying open covering, is given as follows: $K_\circ(\Re, A) := K(\Re, A) \underset{\bullet}{\wedge} (\Re \subseteq \tau)$.

Definition 11. *[11]. Let Ω be the class of all fuzzifying topological spaces. A unary fuzzy predicate $\Gamma \in \Im(\Omega)$, called fuzzifying compactness, is given as follows:*

$(X,\tau) \in \Gamma := (\forall \Re)(K_\circ(\Re, X) \longrightarrow (\exists \wp)((\wp \leq \Re) \wedge K(\wp, A) \underset{\bullet}{\wedge} FF(\wp)))$, *where $\wp \leq \Re$ means that for any $M \in P(X), \wp(M) \leq \Re(M)$.*

Definition 12. *[11]. Let X be a set. A unary fuzzy predicate $fI \in \Im(\Im(P(X)))$, called fuzzifying finite intersection property, is given as follows:*

$fI(\Re) := (\forall \beta)((\beta \leq \Re) \wedge FF(\beta) \longrightarrow (\exists x)(\forall B)((B \in \beta) \to (x \in B)))$.

Lemma 2. *[1]. Let (X,τ) be a fuzzifying topological space. Then*

(1) $\models \tau \subseteq \tau_P$; (2) $\models \digamma \subseteq \digamma_P$, (3) $\models \digamma_P\left(\bigcap_{\lambda\in\Lambda} A_\lambda\right) \geq \bigwedge_{\lambda\in\Lambda} \digamma_P(A_\lambda)$.

Corollary 1. *[1]. $\tau_P(A) = \inf_{x\in A} N_x^P(A)$.*

Theorem 1. *[1]. For any $x, A, B, \models A \subseteq B \to \left(A \in N_x^P \to B \in N_x^P\right)$.*

3 Pre-irresolute functions

Definition 13. *Let (X,τ) and (Y,σ) be two fuzzifying topological spaces and let $f \in Y^X$. A unary fuzzy predicate $I_P \in \Im(Y^X)$, called fuzzifying pre-irresolute, is given as follows: $I_P(f) := \forall B(B \in \sigma_P \to f^{-1}(B) \in \tau_P)$.*

Theorem 2. *Let (X,τ) and (Y,σ) be two fuzzifying topological spaces and let $f \in Y^X$. Then*

$\models f \in I_P \to f \in C_P$.

Proof. From Lemma 2.2 we have $\sigma(B) \leq \sigma_P(B)$ and the result holds. □

Definition 14. *Let (X,τ) and (Y,σ) be two fuzzifying topological spaces and let $f \in Y^X$. We define the unary fuzzy predicates $\alpha_k \in \Im(Y^X)$, where $k = 1, ..., 5$, as follows:*

(1) $f \in \alpha_1 = \forall B\left(B \in \digamma_P^Y \to f^{-1}(B) \in \digamma_P^X\right)$, where $\digamma_P^X$ and $\digamma_P^Y$ are the fuzzifying pre-closed subsets of X and Y, respectively;

(2) $f \in \alpha_2 = \forall x \forall u\left(u \in N_{f(x)}^{P^Y} \to f^{-1}(u) \in N_x^{P^X}\right)$, where N^{P^X} and N^{P^Y} are the family of fuzzifying pre-neighborhood systems of X and Y, respectively;

(3) $f \in \alpha_3 = \forall x \forall u \left(u \in N_{f(x)}^{P^Y} \to \exists v (f(v) \subseteq u \to v \in N_x^{P^X}) \right)$;
(4) $f \in \alpha_4 = \forall A \left(f \left(cl_P^X(A) \right) \subseteq cl_P^Y \left(f(A) \right) \right)$;
(5) $f \in \alpha_5 = \forall B \left(cl_P^X(f^{-1}(B)) \subseteq f^{-1}(cl_P^Y(B)) \right)$.

Theorem 3. $\models\ f \in I_P \leftrightarrow f \in \alpha_k,\ k = 1, ..., 5.$

Proof. (a) We will prove that $\models\ f \in I_P \leftrightarrow f \in \alpha_1$.

$$
\begin{aligned}
[f \in \alpha_1] &= \inf_{B \in P(Y)} \min \left(1, 1 - F_P^Y(B) + F_P^X(f^{-1}(B))\right) \\
&= \inf_{B \in P(Y)} \min \left(1, 1 - \sigma_P(Y - B) + \tau_P(X - f^{-1}(B))\right) \\
&= \inf_{B \in P(Y)} \min \left(1, 1 - \sigma_P(Y - B) + \tau_P(f^{-1}(Y - B))\right) \\
&= \inf_{u \in P(Y)} \min \left(1, 1 - \sigma_P(u) + \tau_P(f^{-1}(u))\right) \\
&= [f \in I_P].
\end{aligned}
$$

(b) We will prove that $\models\ f \in I_P \leftrightarrow f \in \alpha_2$. First, we prove that $[f \in \alpha_2] \geq [f \in I_P]$. If $N_{f(x)}^{P^Y}(u) \leq N_x^{P^X}(f^{-1}(u))$, then $\min \left(1, 1 - N_{f(x)}^{P^Y}(u) + N_x^{P^X}(f^{-1}(u))\right) = 1 \geq [f \in I_P]$. Suppose $N_{f(x)}^{P^Y}(u) > N_x^{P^X}(f^{-1}(u))$. It is clear that, if $f(x) \in A \subseteq u$, then $x \in f^{-1}(A) \subseteq f^{-1}(u)$. Then,

$$
\begin{aligned}
N_{f(x)}^{P^Y}(u) - N_x^{P^X}(f^{-1}(u)) &= \sup_{f(x) \in A \subseteq u} \sigma_P(A) - \sup_{x \in B \subseteq f^{-1}(u)} \tau_P(B) \\
&\leq \sup_{f(x) \in A \subseteq u} \sigma_P(A) - \sup_{f(x) \in A \subseteq u} \tau_P(f^{-1}(A)) \\
&\leq \sup_{f(x) \in A \subseteq u} \left(\sigma_P(A) - \tau_P(f^{-1}(A))\right)
\end{aligned}
$$

So

$$1 - N_{f(x)}^{P^Y}(u) + N_x^{P^X}(f^{-1}(u)) \geq \inf_{f(x) \in A \subseteq u} \left(1 - \sigma_P(A) + \tau_P(f^{-1}(A))\right).$$

Therefore

$$
\begin{aligned}
&\min(1, 1 - N_{f(x)}^{P^Y}(u) + N_x^{P^X}(f^{-1}(u))) \\
&\geq \inf_{f(x) \in A \subseteq u} \min \left(1, 1 - \sigma_P(A) + \tau_P(f^{-1}(A))\right) \\
&\geq \inf_{v \in P(Y)} \min \left(1, 1 - \sigma_P(v) + \tau_P(f^{-1}(v))\right) = [f \in I_P].
\end{aligned}
$$

Hence

$$\inf_{x \in X} \inf_{u \in P(Y)} \min(1, 1 - N_{f(x)}^{P^Y}(u) + N_x^{P^X}(f^{-1}(u))) \geq [f \in I_P].$$

Second, we prove that $[f \in I_P] \geq [f \in \alpha_2]$. From Corollary 2.1, we have

$$
\begin{aligned}
[f \in I_P] &= \inf_{u \in P(Y)} \min \left(1, 1 - \sigma_P(u) + \tau_P(f^{-1}(u))\right) \\
&\geq \inf_{u \in P(Y)} \min(1, 1 - \inf_{f(x) \in u} N_{f(x)}^{P^Y}(u) + \inf_{x \in f^{-1}(u)} N_x^{P^X}(f^{-1}(u))) \\
&\geq \inf_{u \in P(Y)} \min(1, 1 - \inf_{x \in f^{-1}(u)} N_{f(x)}^{P^Y}(u) + \inf_{x \in f^{-1}(u)} N_x^{P^X}(f^{-1}(u))) \\
&\geq \inf_{x \in x} \inf_{u \in P(Y)} \min(1, 1 - N_{f(x)}^{P^Y}(u) + N_x^{P^X}(f^{-1}(u))) = [f \in \alpha_2].
\end{aligned}
$$

(c) We prove that $[f \in \alpha_2] = [f \in \alpha_3]$. From Theorem 2.1 we have

$$[f \in \alpha_3] = \inf_{x \in x} \inf_{u \in P(Y)} \min(1, 1 - N_{f(x)}^{P^Y}(u) + \sup_{v \in P(X), f(v) \subseteq u} N_x^{P^X}(v)))$$

$\geq \inf_{x\in x} \inf_{u\in P(X)} \min(1, 1 - N^{P^Y}_{f(x)}(u) + N^{P^X}_{x}(f^{-1}(u)))$
$[f \in \alpha_2]$.

(d) We prove that $[f \in \alpha_4] = [f \in \alpha_5]$. First, since for any fuzzy set $\widetilde{A}$ we have $\left[f^{-1}(f(\widetilde{A})) \supseteq \widetilde{A}\right] = 1$, then for any $B \in P(Y)$ we have $\left[f^{-1}(f(cl^X_P(f^{-1}(B)))) \supseteq cl^X_P(f^{-1}(B))\right] = 1$. Also, since $\left[f(f^{-1}(B)) \subseteq B\right] = 1$, then we have that $\left[cl^Y_P(f(f^{-1}(B))) \subseteq cl^Y_P(B)\right] = 1$. Then from Lemma 1.2 (2) [10] we have

$$\left[cl^X_P(f^{-1}(B)) \subseteq f^{-1}(cl^Y_P(B))\right] \geq \left[f^{-1}(f(cl^X_P(f^{-1}(B)))) \subseteq f^{-1}(cl^Y_P(B))\right]$$
$$\geq \left[f^{-1}(f(cl^X_P(f^{-1}(B)))) \subseteq f^{-1}(cl^Y_P(f(f^{-1}(B))))\right]$$
$$\geq \left[f(cl^X_P(f^{-1}(B))) \subseteq cl^Y_P(f(f^{-1}(B)))\right].$$

Therefore

$$[f \in \alpha_5] = \inf_{B\in P(Y)} \left[cl^X_P(f^{-1}(B)) \subseteq f^{-1}(cl^Y_P(B))\right]$$
$$\geq \inf_{B\in P(Y)} \left[f(cl^X_P(f^{-1}(B))) \subseteq cl^Y_P(f(f^{-1}(B)))\right]$$
$$\geq \inf_{A\in P(X)} \left[f(cl^X_P(A)) \subseteq cl^Y_P(f(A))\right]$$
$$= [f \in \alpha_4].$$

Second, for each $A \in P(X)$, there exists $B \in P(Y)$ such that $f(A) = B$ and $f^{-1}(B) \supseteq A$. Hence from Lemma 1.2 (1) [10] we have

$$[f \in \alpha_4] = \inf_{A\in P(X)} \left[f(cl^X_P(A)) \subseteq cl^Y_P(f(A))\right]$$
$$\geq \inf_{A\in P(X)} \left[f(cl^X_P(A)) \subseteq f(f^{-1}(cl^Y_P(f(A))))\right]$$
$$\geq \inf_{A\in P(X)} \left[cl^X_P(A) \subseteq f^{-1}(cl^Y_P(f(A)))\right]$$
$$\geq \inf_{B\in P(Y), B=f(A)} \left[cl^X_P(f^{-1}(B)) \subseteq f^{-1}(cl^Y_P(B))\right]$$
$$\geq \inf_{B\in P(Y)} \left[cl^X_P(f^{-1}(B)) \subseteq f^{-1}(cl^Y_P(B))\right]$$
$$= [f \in \alpha_5].$$

(e) We want to prove that $\models f \in \alpha_2 \leftrightarrow f \in \alpha_5$.

$$[f \in \alpha_5] = \inf_{B\in P(Y)} \left[cl^X_P(f^{-1}(B)) \subseteq f^{-1}(cl^Y_P(B))\right]$$
$$= \inf_{B\in P(Y)} \inf_{x\in X} \min\left(1, 1 - (1 - N^{P^X}_{x}(X - f^{-1}(B))) + 1 - N^{P^Y}_{f(x)}(Y - B)\right)$$
$$= \inf_{B\in P(Y)} \inf_{x\in X} \min\left(1, 1 - N^{P^Y}_{f(x)}(Y - B) + N^{P^X}_{x}(f^{-1}(Y - B))\right)$$
$$= \inf_{B\in P(Y)} \inf_{x\in X} \min\left(1, 1 - N^{P^Y}_{f(x)}(u) + N^{P^X}_{x}(f^{-1}(u))\right)$$
$$= [f \in \alpha_2].$$
□

4 Strong compactness in fuzzifying topology

Definition 15. *A fuzzifying topological space (X, τ) is said to be p-fuzzifying topological space if $\tau_P(A \cap B) \geq \tau_P(A) \wedge \tau_P(B)$.*

Definition 16. *A binary fuzzy predicate $K_P \in \Im(\Im(P(X)) \times P(X))$, called fuzzifying pre-open covering, is given as $K_P(\Re, A) := K(\Re, A) \underset{\bullet}{\wedge} (\Re \subseteq \tau_P)$.*

Definition 17. *Let Ω be the class of all fuzzifying topological spaces. A unary fuzzy predicate $\Gamma_P \in \Im(\Omega)$, called fuzzifying strong compactness, is given as follows:*

(1) $(X,\tau) \in \Gamma_P := (\forall \Re)(K_P(\Re, X) \longrightarrow (\exists \wp)((\wp \le \Re) \wedge K(\wp, X) \dot\wedge FF(\wp)))$;

(2) If $A \subseteq X$, *then* $\Gamma_P(A) := \Gamma_P(A, \tau/_A)$.

Lemma 3. $\vDash K_\circ(\Re, A) \longrightarrow K_P(\Re, A)$.

Proof. Since from Lemma 2.2 $\vDash \tau \subseteq \tau_P$, then we have $[\Re \subseteq \tau] \le [\Re \subseteq \tau_P]$. So, $[K_\circ(\Re, A)] \le [K_P(\Re, A)]$. □

Theorem 4. $\vDash (X,\tau) \in \Gamma_P \longrightarrow (X,\tau) \in \Gamma$.

Proof. From Lemma 4.1 the proof is immediate. □

Theorem 5. *For any fuzzifying topological space (X,τ) and $A \subseteq X$,*

$\Gamma_P(A) \longleftrightarrow (\forall \Re)(K_P(\Re, A) \longrightarrow (\exists \wp)((\wp \le \Re) \wedge K(\wp, A) \dot\wedge FF(\wp)))$, *where K_P is related to τ.*

Proof. For any $\Re \in \Im(\Im(X))$, we set $\overline{\Re} \in \Im(\Im(A))$ defined as $\overline{\Re}(C) = \Re(B)$ with $C = A \cap B, B \subseteq X$. Then

$K(\overline{\Re}, A) = \inf\limits_{x\in A} \sup\limits_{x \in C} \overline{\Re}(C) = \inf\limits_{x\in A} \sup\limits_{x\in C = A\cap B} \Re(B) = \inf\limits_{x\in A} \sup\limits_{x\in B} \Re(B) = K(\Re, A)$, because $x \in A$ and $x \in B$ if and only if $x \in A \cap B$. Therefore

$[\overline{\Re} \subseteq \tau_P/_A] = \inf\limits_{C\subseteq A} \min(1, 1 - \overline{\Re}(C) + \tau_P/_A(C))$

$= \inf\limits_{C\subseteq A} \min(1, 1 - \sup\limits_{C=A\cap B, B\subseteq X} \Re(B) + \sup\limits_{C=A\cap B, B\subseteq X} \tau_P(B))$

$\ge \inf\limits_{C\subseteq A, C=A\cap B, B\subseteq X} \min(1, 1 - \Re(B) + \tau_P(B))$

$\ge \inf\limits_{B\subseteq X} \min(1, 1 - \Re(B) + \tau_P(B)) = [\Re \subseteq \tau_P]$.

For any $\wp \le \overline{\Re}$, we define $\wp' \in \Im(P(X))$ as follows:

$\wp'(B) = \begin{cases} \wp(B) & \text{if } B\subseteq A, \\ 0 & \text{otherwise.} \end{cases}$

Then $\wp' \le \Re, FF(\wp') = FF(\wp)$ and $K(\wp', A) = K(\wp, A)$.

Furthermore, we have

$[\Gamma_P(A) \dot\wedge K_P(\Re, A)] \le [\Gamma_P(A) \dot\wedge K'_P(\overline{\Re}, A)]$

$\le [(\exists \wp)((\wp \le \overline{\Re}) \wedge K(\wp, A) \dot\wedge FF(\wp))]$

$\le [(\exists \wp')((\wp' \le \Re) \wedge K(\wp', A) \dot\wedge FF(\wp'))]$

$\le [(\exists \text{ß})((\text{ß} \le \Re) \wedge K(\text{ß}, A) \dot\wedge FF(\text{ß}))]$.

Then $\Gamma_P(A) \le [K_P(\Re, A)] \longrightarrow [(\exists \text{ß})((\text{ß} \le \Re) \wedge K(\text{ß}, A) \dot\wedge FF(\text{ß}))]$,

where $K'_P(\overline{\Re}, A) = [K(\overline{\Re}, A) \dot\wedge (\overline{\Re} \subseteq \tau_P/_A)]$. Therefore

$\Gamma_P(A) \le \inf\limits_{\Re\in\Im(P(X))} [K_P(\Re, A) \longrightarrow (\exists \text{ß})((\text{ß} \le \Re) \wedge K(\text{ß}, A) \dot\wedge FF(\text{ß}))]$

$= [(\forall \Re)(K_P(\Re, A) \longrightarrow (\exists \text{ß})((\text{ß} \le \Re) \wedge K(\text{ß}, A) \dot\wedge FF(\text{ß})))]$.

Conversely, for any $\Re \in \Im(P(A))$, if $[\Re \subseteq \tau_P/_A] = \inf\limits_{B\subseteq A} \min(1, 1 - \Re(B) + \tau_P/_A(B)) = \lambda$, then for any $n \in N$ and $B \subseteq A$, $\sup\limits_{B=A\cap C, C\subseteq X} \tau_P(C) =$

$\tau_P/_A(B) > \lambda + \Re(B) - 1 - \frac{1}{n}$, and there exists $C_B \subseteq X$ such that $C_B \cap A = B$ and $\tau_P(C_B) > \lambda + \Re(B) - 1 - \frac{1}{n}$. Now, we define $\overline{\Re} \in \Im(P(X))$ as $\overline{\Re}(C) = \max_{B\subseteq A}(0, \lambda + \Re(B) - 1 - \frac{1}{n})$. Then $[\overline{\Re} \subseteq \tau_P] = 1$ and

$$\begin{aligned} K(\overline{\Re}, A) &= \inf_{x\in A} \sup_{x\in C\subseteq X} \overline{\Re}(C) \\ &= \inf_{x\in A} \sup_{x\in B} \overline{\Re}(C_B) \\ &\geq \inf_{x\in A} \sup_{x\in B} (\lambda + \Re(B) - 1 - \tfrac{1}{n}) \\ &= \inf_{x\in A} \sup_{x\in B} \Re(B) + \lambda - 1 - \tfrac{1}{n} \\ &= K(\Re, A) + \lambda - 1 - \tfrac{1}{n}, \end{aligned}$$

$$\begin{aligned} K_P(\overline{\Re}, A) &= [K(\overline{\Re}, A) \dot{\wedge} (\overline{\Re} \subseteq \tau_P)] \\ &= [K(\overline{\Re}, A)] \geq \max(0, K(\Re, A) + \lambda - 1 - \tfrac{1}{n}) \\ &\geq \max(0, K(\Re, A) + \lambda - 1) - \tfrac{1}{n} = K_P(\Re', A) - \tfrac{1}{n}. \end{aligned}$$

For any $\wp \leq \overline{\Re}$, we set $\wp' \in \Im(P(A))$ as $\wp'(B) = \wp(C_B), B \subseteq A$. Then $\wp' \leq \Re, FF(\wp') = FF(\wp)$ and $K(\wp', A) = K(\wp, A)$. Therefore

$$\begin{aligned} &[(\forall\Re)(K_P(\Re, A) \longrightarrow (\exists\wp)((\wp \leq \Re) \wedge K(\wp, A) \dot{\wedge} FF(\wp))] \dot{\wedge} \\ &[K'_P(\Re, A)] - \tfrac{1}{n} \\ &\leq [(\forall\Re)(K_P(\Re, A) \longrightarrow (\exists\wp)((\wp \leq \Re) \wedge K(\wp, A) \dot{\wedge} FF(\wp))] \dot{\wedge} \\ &\left([K'_P(\Re, A)] - \tfrac{1}{n}\right) \\ &\leq [K_P(\overline{\Re}, A) \longrightarrow (\exists\wp)((\wp \leq \overline{\Re}) \wedge K(\wp, A) \dot{\wedge} FF(\wp))] \dot{\wedge} [K_P(\overline{\Re}, A)] \\ &\leq [(\exists\wp)((\wp \leq \overline{\Re}) \wedge K(\wp, A) \dot{\wedge} FF(\wp))] \\ &\leq [(\exists\wp')((\wp' \leq \Re) \wedge K(\wp', A) \dot{\wedge} FF(\wp'))] \\ &\leq [(\exists ß)((ß \leq \Re) \wedge K(ß, A) \dot{\wedge} FF(ß))]. \end{aligned}$$

Let $n \longrightarrow \infty$. We obtain

$$\begin{aligned} &[(\forall\Re)(K_P(\Re, A) \longrightarrow (\exists\wp)((\wp \leq \Re) \wedge K(\wp, A) \dot{\wedge} FF(\wp)))] \dot{\wedge} \\ &[K'_P(\Re, A)] \leq [(\exists ß)((ß \leq \Re) \wedge K(ß, A) \dot{\wedge} FF(ß))]. \end{aligned}$$

Then

$$\begin{aligned} &[(\forall\Re)(K_P(\Re, A) \longrightarrow (\exists\wp)((\wp \leq \Re) \wedge K(\wp, A) \dot{\wedge} FF(\wp))] \\ &\leq [K'_P(\Re, A) \longrightarrow (\exists ß)((ß \leq \Re) \wedge K(ß, A) \dot{\wedge} FF(ß))] \\ &\leq \inf_{\Re\in\Im(P(X))} [K'_P(\Re, A) \longrightarrow (\exists ß)((ß \leq \Re) \wedge K(ß, A) \dot{\wedge} FF(ß))] \\ &= \Gamma_P(A). \end{aligned}$$

□

Theorem 6. *Let (X, τ) be a fuzzifying topological space.*
$\pi_1 := (\forall\Re)((\Re \in \Im(P(X))) \wedge (\Re \subseteq F_P) \dot{\wedge} fI(\Re) \longrightarrow (\exists x)(\forall A)(A \in \Re \to x \in A));$
$\pi_2 := (\forall\Re)(\exists B)(((\Re \subseteq F_P) \wedge (B \in \tau_P)) \dot{\wedge} (\forall\wp)((\wp \leq \Re) \dot{\wedge} FF(\wp) \to \neg(\bigcap \wp \subseteq B)) \to \neg(\bigcap \Re \subseteq B))$. *Then* $\models \Gamma_P(X, \tau) \longleftrightarrow \pi_i$, $i = 1, 2$.

Proof. (a) We prove $\Gamma_P(X, \tau) = [\pi_1]$. For any $\Re \in \Im(P(X))$, we set $\Re^c(X - A) = \Re(A)$. Then

$[\Re \subseteq \tau_P] = \inf_{A \in P(X)} \min(1, 1 - \Re(A) + \tau_P(A))$
$= \inf_{X-A \in P(X)} \min(1, 1 - \Re^c(X-A) + \digamma_P(X-A)) = [\Re^c \subseteq \digamma_P]$,
$FF(\Re) = 1 - \inf\{\alpha \in [0,1] : F(\Re_\alpha)\} = 1 - \inf\{\alpha \in [0,1] : F(\Re^c_\alpha)\} = FF(\Re^c)$
and
$ß \leq \Re^c \Longleftrightarrow ß(M) \leq \Re^c(M) \Longleftrightarrow ß^c(X-M) \leq \Re(X-M) \Longleftrightarrow ß^c \leq \Re$.
Therefore
$\Gamma_P(X,\tau) = [(\forall \Re)(K_P(\Re, X) \longrightarrow (\exists \wp)((\wp \leq \Re) \wedge K(\wp, X) \dot{\wedge} FF(\wp)))]$
$= [(\forall \Re)\Big((\Re \subseteq \tau_P) \dot{\wedge} K(\Re, X) \longrightarrow (\exists \wp)((\wp \leq \Re) \wedge K(\wp, X) \dot{\wedge} FF(\wp))\Big)]$
$= [(\forall \Re)\Big((\Re \subseteq \tau_P) \longrightarrow (K(\Re, X) \longrightarrow (\exists \wp)((\wp \leq \Re) \wedge K(\wp, X) \dot{\wedge} FF(\wp)))\Big)]$
$= [(\forall \Re)((\Re^c \subseteq \digamma_P) \longrightarrow ((\forall x)(\exists A)(A \in \Re \wedge x \in A) \longrightarrow$
$(\exists \wp)((\wp \leq \Re) \wedge K(\wp, X) \dot{\wedge} FF(\wp))))]$
$= [(\forall \Re)((\Re^c \subseteq \digamma_P) \longrightarrow ((\forall x)(\exists A)(A \in \Re \wedge x \in A) \longrightarrow$
$(\exists ß^c)((ß^c \leq \Re) \wedge K(ß^c, X) \dot{\wedge} FF(ß^c))))]$
$= [(\forall \Re)((\Re^c \subseteq \digamma_P) \longrightarrow ((\forall x)(\exists A)(A \in \Re \wedge x \in A) \longrightarrow$
$(\exists ß)((ß \leq \Re^c) \wedge FF(ß) \dot{\wedge} K(ß^c, X))))]$
$= [(\forall \Re)(\Re^c \subseteq \digamma_P \longrightarrow ((\forall x)(\exists A)(A \in \Re \wedge x \in A) \longrightarrow$
$(\exists ß)((ß \leq \Re^c) \wedge FF(ß) \dot{\wedge} (\forall x)(\exists B)(B \in ß^c \wedge x \in B))]$
$= [(\forall \Re)(\Re^c \subseteq \digamma_P \longrightarrow (\neg((\exists ß)(ß \leq \Re^c \wedge FF(ß) \dot{\wedge}$
$(\forall x)(\exists B)(B \in ß^c \wedge x \in B)) \rightarrow \ \neg((\forall x)(\exists A)(A \in \Re \wedge x \in A))))]$
$= [(\forall \Re)((\Re^c \subseteq \digamma_P) \longrightarrow (\text{fI}(\Re^c) \longrightarrow \neg((\forall x)(\exists A)(A \in \Re \wedge x \in A)))]$
$= [(\forall \Re)((\Re^c \subseteq \digamma_P) \dot{\wedge} \text{fI}(\Re^c) \longrightarrow (\exists x)(\forall A)(A \in \Re^c \rightarrow x \in A))] = [\pi_1]$.

(b) We prove $[\pi_1] = [\pi_2]$. Let $X - B \in P(X)$. For any $\Re \in \Im(P(X))$.
$[(\Re \subseteq \digamma_P) \wedge (B \in \tau_P)] = [(\Re \subseteq \digamma_P) \wedge (X - B \in \digamma_P)]$
$= \inf_{A \in P(X)} \min(1, 1 - \Re(A) + \digamma_P(A)) \wedge \digamma_P(X - B)$
$= \inf_{A \in P(X)} \min(1, 1 - \Re(A) + \digamma_P(A)) \wedge$
$\inf_{A \in P(X)} \min(1, 1 - [A \in \{X - B\}] + \digamma_P(A))$
$= \inf_{A \in P(X)} \min(1, 1 - [(\Re \cup \{X - B\})(A)] + \digamma_P(A))$
$= [(\Re \cup \{X - B\}) \subseteq \digamma_P]$.
Therefore, for any $ß \in \Im(P(X))$, let $\wp = ß \backslash \{X - B\} \in \Im(P(X))$.
$\wp(A) = \begin{cases} ß(A), & A \neq X - B \\ 0, & A = X - B \end{cases}$.
Then $\wp \leq ß$, $\wp \cup \{X - B\} \geq ß$, $[FF(\wp)] = [FF(ß)]$,
$[\wp \leq \Re] = [ß \leq (\Re \cup \{X - B\})]$ and
$[(\forall \wp)((\wp \leq \Re) \dot{\wedge} FF(\wp) \rightarrow (\exists x)(\forall A)(A \in (\wp \cup \{X - B\}) \rightarrow$
$(x \in A)))]$
$= \inf_{\wp \leq \Re} \min(1, 1 - [FF(\wp)] + \sup_{x \in X} \inf_{A \in P(X)} ((\wp \cup \{X - B\})(A) \rightarrow A(x)))$

$\leq \inf_{ß \leq (\Re \cup \{X - B\})} \min(1, 1 - [FF(ß)] + \sup_{x \in X} \inf_{A \in P(X)} (ß(A) \rightarrow A(x)))$

$=$fI$(\Re \cup \{X - B\})$. Furthermore, we have
$\pi_1 \underset{\bullet}{\wedge} [((\Re \subseteq \mathcal{F}_P) \wedge (B \in \tau_P)) \underset{\bullet}{\wedge} (\forall \wp)\,((\wp \leq \Re) \underset{\bullet}{\wedge} FF(\wp) \rightarrow$
$\neg\,(\bigcap \wp \subseteq B))]$
$= \pi_1 \underset{\bullet}{\wedge} [(\Re \cup \{X - B\} \subseteq \mathcal{F}_P) \underset{\bullet}{\wedge} (\forall \wp)\,((\wp \leq \Re) \underset{\bullet}{\wedge} FF(\wp) \rightarrow$
$(\exists x)(\forall A)\,(A \in (\wp \cup \{X - B\}) \rightarrow x \in A))]$
$= \pi_1 \underset{\bullet}{\wedge} [(\Re \cup \{X - B\} \subseteq \mathcal{F}_P) \underset{\bullet}{\wedge}fI(\Re \cup \{X - B\})]$
$\leq [(\exists x)(\forall A)\,(A \in (\Re \cup \{X - B\}) \rightarrow x \in A)]$
$= [\neg\,(\bigcap \Re \subseteq B)]$. Therefore
$\pi_1 \leq \inf\limits_{\Re \in \Im(P(X))} \sup\limits_{B \subseteq X} ((\Re \subseteq \mathcal{F}_P \wedge B \in \tau_P) \underset{\bullet}{\wedge} (\forall \wp)\,((\wp \leq \Re) \underset{\bullet}{\wedge} FF(\wp) \rightarrow$
$\neg\,(\bigcap \wp \subseteq B)) \rightarrow \neg\,(\bigcap \Re \subseteq B))$
$= \pi_2$. Conversely,
$\pi_2 \underset{\bullet}{\wedge} [(\Re \subseteq \mathcal{F}_P) \underset{\bullet}{\wedge}fI(\Re)] = \pi_2 \underset{\bullet}{\wedge} [((\Re \backslash \{B\}) \cup \{B\}) \subseteq \mathcal{F}_P] \underset{\bullet}{\wedge}$
$[$fI$\,((\Re \backslash \{B\}) \cup \{B\})]$
$= \pi_2 \underset{\bullet}{\wedge} [(\Re' \subseteq \mathcal{F}_P) \wedge (X - B \in \tau_P) \underset{\bullet}{\wedge} (\forall \wp)\,((\wp \leq \Re') \underset{\bullet}{\wedge} FF(\wp) \rightarrow$
$(\exists x)(\forall A)\,(A \in (\wp \cup \{B\}) \rightarrow x \in A))]$
$= \pi_2 \underset{\bullet}{\wedge} [(\Re' \subseteq \mathcal{F}_P) \wedge (X - B \in \tau_P) \underset{\bullet}{\wedge} (\forall \wp)\,((\wp \leq \Re') \underset{\bullet}{\wedge} FF(\wp) \rightarrow$
$\neg\,(\bigcap \wp \subseteq X - B))]$
$\leq [\neg\,(\bigcap \Re' \subseteq X - B)] = [(\exists x)(\forall A)((A \in (\Re' \cup \{B\}) \rightarrow (x \in A))]$
$= [(\exists x)(\forall A)(A \in \Re \rightarrow (x \in A))]$. Therefore
$\pi_2 \leq \inf\limits_{\Re \in \Im(P(X))} [(\Re \subseteq \mathcal{F}_P) \underset{\bullet}{\wedge}fI(\Re) \rightarrow (\exists x)(\forall A)(A \in \Re \rightarrow (x \in A))] = \pi_1$. □

5 Some properties of fuzzifying strong compactness

Theorem 7. *For any fuzzifying topological space* (X, τ) *and* $A \subseteq X$*,*
$\models \Gamma_P(X, \tau) \underset{\bullet}{\wedge} A \in \mathcal{F}_P \rightarrow \Gamma_P(A)$.

Proof. For any $\Re \in \Im(P(A))$, we define $\overline{\Re} \in \Im(P(X))$ as follows:
$\overline{\Re}(B) = \begin{cases} \Re(B) & \text{if } B \subseteq A, \\ 0 & \text{otherwise.} \end{cases}$
Then $FF(\overline{\Re}) = 1 - \inf\left\{\alpha \in [0,1] : F(\overline{\Re}_\alpha)\right\}$
$= 1 - \inf\{\alpha \in [0,1] : F(\Re_\alpha)\} = FF(\Re)$ and
$\sup\limits_{x \in X} \inf\limits_{x \notin B \subseteq X} \left(1 - \overline{\Re}(B)\right) =$
$\sup\limits_{x \in X} \left(\left(\inf\limits_{x \notin B \subseteq A} \left(1 - \overline{\Re}(B)\right)\right) \wedge \left(\inf\limits_{x \notin B \subsetneq A} \left(1 - \overline{\Re}(B)\right)\right)\right)$
$= \sup\limits_{x \in X} \left(\inf\limits_{x \notin B \subseteq A} \left(1 - \overline{\Re}(B)\right)\right) \wedge \sup\limits_{x \in X} \left(\inf\limits_{x \notin B \subsetneq A} \left(1 - \overline{\Re}(B)\right)\right)$
$= \sup\limits_{x \in X} \left(\inf\limits_{x \notin B \subseteq A} (1 - \Re(B))\right)$
$= \sup\limits_{x \in A} \left(\inf\limits_{x \notin B \subseteq A} (1 - \Re(B))\right) \vee \sup\limits_{x \notin A} \left(\inf\limits_{x \notin B \subseteq A} (1 - \Re(B))\right)$

If $x \notin A$, then for any $x' \in A$ we have
$\inf_{x\notin B\subseteq A}(1-\Re(B)) = \inf_{B\subseteq A}(1-\Re(B)) \leq \inf_{x'\notin B\subseteq A}(1-\Re(B))$.
Therefore, $\sup_{x\in X}\inf_{x\notin B\subseteq X}\left(1-\overline{\Re}(B)\right) = \sup_{x\in A}\inf_{x\notin B\subseteq A}(1-\Re(B))$,
$[\text{fI}(\overline{\Re})] = [(\forall \text{ß})((\text{ß} \leq \overline{\Re}) \wedge FF(\text{ß}) \longrightarrow (\exists x)(\forall B)((B\in\overline{\Re}) \rightarrow (x\in B)))]$

$$= \inf_{\overline{\text{ß}}\leq\overline{\Re}} \min\left(1, 1-FF(\overline{\text{ß}}) + \sup_{x\in X}\inf_{x\notin B\subseteq X}\left(1-\overline{\Re}(B)\right)\right)$$

$$= \inf_{\text{ß}\leq\Re}\min\left(1, 1-FF(\text{ß}) + \sup_{x\in A}\inf_{x\notin B\subseteq A}(1-\Re(B))\right) = [\text{fI}(\Re)].$$

We want to prove that $\digamma_P(A) \underset{\bullet}{\wedge} [\Re \subseteq \digamma_P/_A] \leq [\overline{\Re} \subseteq \digamma_P]$.
In fact, from Lemma 2.2 (3) we have

$$\digamma_P(A) \underset{\bullet}{\wedge} [\Re \subseteq \digamma_P/_A] =$$

$$\max\left(0, \digamma_P(A) + \inf_{B\subseteq A}\min(1, 1-\Re(B) + \digamma_P/_A(B)) - 1\right)$$

$$\leq \inf_{B\subseteq A}(1-\Re(B)) + (\digamma_P(A) + \digamma_P/_A(B) - 1)$$

$$\leq \inf_{B\subseteq A}(1-\Re(B)) + (\digamma_P(A) \wedge \digamma_P/_A(B))$$

$$= \inf_{B\subseteq A}(1-\Re(B)) + \left(\digamma_P(A) \wedge \sup_{B'\cap A=B, B'\subseteq X}\digamma_P(B')\right)$$

$$= \inf_{B\subseteq A}(1-\Re(B)) + \sup_{B'\cap A=B, B'\subseteq X}(\digamma_P(A) \wedge \digamma_P(B'))$$

$$\leq \inf_{B\subseteq A}(1-\Re(B)) + \sup_{B'\cap A=B, B'\subseteq X}(\digamma_P(A\cap B'))$$

$$\leq \inf_{B\subseteq A}(1-\Re(B)) + \digamma_P(B)$$

$$= \inf_{B\subseteq A}\min(1, 1-\Re(B) + \digamma_P(B))$$

$$\leq \inf_{B\subseteq A}\min\left(1, 1-\overline{\Re}(B) + \digamma_P(B)\right) = [\overline{\Re} \subseteq \digamma_P].$$

Furthermore, from Theorem 4.3 we have

$$\Gamma_P(X,\tau) \underset{\bullet}{\wedge} \digamma_P(A) \underset{\bullet}{\wedge} [\Re \subseteq \digamma_P/_A] \underset{\bullet}{\wedge} \text{fI}(\Re)$$

$$\leq \Gamma_P(X,\tau) \underset{\bullet}{\wedge} [\overline{\Re} \subseteq \digamma_P] \underset{\bullet}{\wedge} \text{fI}(\overline{\Re})$$

$$\leq \sup_{x\in X}\inf_{x\notin B\subseteq A}\left(1-\overline{\Re}(B)\right) = \sup_{x\in A}\inf_{x\notin B\subseteq A}(1-\Re(B)).$$ Then

$$\Gamma_P(X,\tau) \underset{\bullet}{\wedge} \digamma_P(A) \leq [\Re \subseteq \digamma_P/_A] \underset{\bullet}{\wedge} \text{fI}(\Re) \longrightarrow \sup_{x\in A}\inf_{x\notin B\subseteq A}(1-\Re(B))$$

$$\leq \inf_{\Re\in\Im(P(A))}\left([\Re \subseteq \digamma_P/_A] \underset{\bullet}{\wedge} \text{fI}(\Re) \longrightarrow \sup_{x\in A}\inf_{x\notin B\subseteq A}(1-\Re(B))\right)$$

$$= \Gamma_P(A).$$ □

Theorem 8. *Let* (X,τ) *and* (Y,σ) *be any two fuzzifying topological space and* $f \in Y^X$ *is surjection. Then* $\models \Gamma_P(X,\tau) \underset{\bullet}{\wedge} C_P(f) \longrightarrow \Gamma(f(X))$.

Proof. For any ß$\in \Im(P(Y))$, we define $\Re \in \Im(P(X))$ as follows:
$\Re(A) = f^{-1}(\text{ß})(A) = \text{ß}(f(A))$. Then
$K(\Re, X) = \inf_{x\in X}\sup_{x\in A}\Re(A) = \inf_{x\in X}\sup_{x\in A}\text{ß}(f(A))$

$$= \inf_{x\in X} \sup_{f(x)\in B} \text{ß}(B) = \inf_{y\in f(X)} \sup_{y\in B} \text{ß}(B) = K(\text{ß}, f(X)),$$

$$[\text{ß} \subseteq \sigma] \underset{\bullet}{\wedge} [C_P(f)] =$$
$$\inf_{B\subseteq Y} \min(1, 1 - \text{ß}(B) + \sigma(B)) \underset{\bullet}{\wedge} \inf_{B\subseteq Y} \min\left(1, 1 - \sigma(B) + \tau_P\left(f^{-1}(B)\right)\right)$$
$$= \max(0, \inf_{B\subseteq Y} \min(1, 1 - \text{ß}(B) + \sigma(B)) +$$
$$\inf_{B\subseteq Y} \min\left(1, 1 - \sigma(B) + \tau_P\left(f^{-1}(B)\right)\right) - 1)$$
$$\leq \inf_{B\subseteq Y} \max(0, \min(1, 1 - \text{ß}(B) + \sigma(B)) +$$
$$\min\left(1, 1 - \sigma(B) + \tau_P\left(f^{-1}(B)\right)\right) - 1)$$
$$\leq \inf_{B\subseteq Y} \min\left(1, 1 - \text{ß}(B) + \tau_P\left(f^{-1}(B)\right)\right)$$
$$= \inf_{A\subseteq X} \inf_{f^{-1}(B)=A} \min\left(1, 1 - \text{ß}(B) + \tau_P\left(f^{-1}(B)\right)\right)$$
$$= \inf_{A\subseteq X} \inf_{f^{-1}(B)=A} \min(1, 1 - \text{ß}(B) + \tau_P(A))$$
$$= \inf_{A\subseteq X} \min(1, 1 - \sup_{f^{-1}(B)=A} \text{ß}(B) + \tau_P(A))$$
$$= \inf_{A\subseteq X} \min(1, 1 - \Re(A) + \tau_P(A)) = [\Re \subseteq \tau_P] .$$

For any $\wp \leq \Re$, we set $\overline{\wp} \in \Im(P(Y))$ defined as follows:
$\overline{\wp}(f(A)) = f(\wp)(f(A)) = \wp(A),\ A \subseteq X$.
Then $\overline{\wp}(f(A)) = f(\wp)(f(A)) \leq f(\Re)(f(A)) = f(f^{-1}(\text{ß})(f(A)) \leq \text{ß}(f(A))$,
$FF(\wp) = 1 - \inf\{\alpha \in [0,1] : F(\wp_{[\alpha]})\} = 1 - \inf\{\alpha \in [0,1] : F(f(\wp)_{[\alpha]})\}$
$= FF(f(\wp) \leq FF(\overline{\wp})$ and

$$K(\overline{\wp}, f(X)) = \inf_{y\in f(X)} \sup_{y\in B} \overline{\wp}(B) = \inf_{y\in f(X)} \sup_{y\in B=f(A)} \wp(A)$$
$$\geq \inf_{y\in f(X)} \sup_{f^{-1}(y)\in A} \wp(A) = \inf_{x\in X} \sup_{x\in A} \wp(A) = K(\wp, X).$$ Furthermore

$$[\Gamma_P(X,\tau)] \underset{\bullet}{\wedge} [C_P(f)] \underset{\bullet}{\wedge} [K'_\circ(\text{ß}, f(X))]$$
$$= [\Gamma_P(X,\tau)] \underset{\bullet}{\wedge} [C_P(f)] \underset{\bullet}{\wedge} [K(\text{ß}, f(X))] \underset{\bullet}{\wedge} [\text{ß} \subseteq \sigma]$$
$$\leq [\Gamma_P(X,\tau)] \underset{\bullet}{\wedge} [\Re \subseteq \tau_P] \underset{\bullet}{\wedge} [K(\Re, X)]$$
$$= [\Gamma_P(X,\tau)] \underset{\bullet}{\wedge} [K_P(\Re, X)]$$
$$\leq [(\exists \wp)((\wp \leq \Re) \wedge K(\wp, X) \underset{\bullet}{\wedge} FF(\wp))]$$
$$\leq [(\exists \wp)((\wp \leq \Re) \wedge K(\overline{\wp}, f(X)) \underset{\bullet}{\wedge} FF(\overline{\wp}))]$$
$$\leq [(\exists \wp')((\wp' \leq \Re) \wedge K(\wp', f(X)) \underset{\bullet}{\wedge} FF(\wp'))],$$ where $K'_\circ$ is related to σ.

Therefore from Theorem 4.2 we obtain

$$[\Gamma_P(X,\tau)] \underset{\bullet}{\wedge} [C_P(f)]$$
$$\leq K'_\circ(\text{ß}, f(X)) \longrightarrow (\exists \wp')((\wp' \leq \Re) \wedge K(\wp', f(X)) \underset{\bullet}{\wedge} FF(\wp'))$$
$$\leq \inf_{\text{ß}\in\Im(P(X))} \left(K'_\circ(\text{ß}, f(X)) \longrightarrow (\exists \wp')((\wp' \leq \Re) \wedge K(\wp', f(X)) \underset{\bullet}{\wedge} FF(\wp'))\right)$$
$$= [\Gamma(f(X))].$$ □

Theorem 9. *Let* (X,τ) *and* (Y,σ) *be any two fuzzifying topological space and* $f \in Y^X$ *is surjection.*

$\models \Gamma_P(X,\tau) \underset{\bullet}{\wedge} I_P(f) \longrightarrow \Gamma_P(f(X))$.

Proof. From the proof of Theorem 5.2 we have for any $\text{ß} \in \Im(P(Y))$ we define $\Re \in \Im(P(X))$ as

$\Re(A) = f^{-1}(\text{ß})(A) = \text{ß}(f(A))$.

Then $K(\Re, X) = K(\text{ß}, f(X))$ and $[\text{ß} \subseteq \sigma_P] \underset{\bullet}{\wedge} [I_P(f)] \leq [\Re \subseteq \tau_P]$. For any $\wp \leq \Re$, we set $\overline{\wp} \in \Im(P(Y))$ defined as $\overline{\wp}(f(A)) = f(\wp)(f(A)) = \wp(A), A \subseteq X$ and we have $FF(\wp) \leq FF(\overline{\wp})$, $K(\overline{\wp}, f(X)) \geq K(\wp, X)$. Therefore

$[\Gamma_P(X,\tau)] \underset{\bullet}{\wedge} [I_P(f)] \underset{\bullet}{\wedge} [K'_P(\text{ß}, f(X))]$

$=[\Gamma_P(X,\tau)] \underset{\bullet}{\wedge} [I_P(f)] \underset{\bullet}{\wedge} [K(\text{ß}, f(X))] \underset{\bullet}{\wedge} [\text{ß} \subseteq \sigma_P]$

$\leq [\Gamma_P(X,\tau)] \underset{\bullet}{\wedge} [\Re \subseteq \tau_P] \underset{\bullet}{\wedge} [K(\Re, X)]$

$=[\Gamma_P(X,\tau)] \underset{\bullet}{\wedge} [K_P(\Re, X)]$

$\leq [(\exists \wp)((\wp \leq \Re) \wedge K(\wp, X) \underset{\bullet}{\wedge} FF(\wp))]$

$\leq [(\exists \wp)((\wp \leq \Re) \wedge K(\overline{\wp}, f(X)) \underset{\bullet}{\wedge} FF(\overline{\wp}))]$

$\leq [(\exists \wp')((\wp' \leq \text{ß}) \wedge K(\wp', f(X)) \underset{\bullet}{\wedge} FF(\wp'))]$, where $K'_P$is related to σ.

Therefore. from Theorem 4.2 we obtain

$[\Gamma_P(X,\tau)] \underset{\bullet}{\wedge} [I(f)]$

$\leq K'_P(\text{ß}, f(X)) \longrightarrow (\exists \wp')((\wp' \leq \text{ß}) \wedge K(\wp', f(X)) \underset{\bullet}{\wedge} FF(\wp'))$

$\leq \inf_{\text{ß} \in \Im(P(X))} \Big(K'_P(\text{ß}, f(X)) \longrightarrow (\exists \wp')((\wp' \leq \text{ß}) \wedge K(\wp', f(X)) \underset{\bullet}{\wedge} FF(\wp')) \Big)$

$= [\Gamma_P(f(X))]$. □

Theorem 10. *Let (X,τ) be any fuzzifying p−topological space and $A, B \subseteq X$. Then*

(1) $T_2^P(X,\tau) \underset{\bullet}{\wedge} (\Gamma_P(A) \wedge \Gamma_P(B)) \wedge A \cap B = \phi \quad \models^{ws} T_2^P(X,\tau) \longrightarrow$
$(\exists U)(\exists V)((U \in \tau_P) \wedge (V \in \tau_P) \wedge (A \subseteq U) \wedge (B \subseteq V) \wedge (U \cap V = \phi))$*;*

(2) $T_2^P(X,\tau) \underset{\bullet}{\wedge} \Gamma_P(A) \quad \models^{ws} T_2^P(X,\tau) \longrightarrow A \in \digamma_P$.

Proof. (1) Assume $A \cap B = \phi$ and $T_2^P(X,\tau) = t$. Let $x \in A$. Then for any $y \in B$ and $\lambda < t$, we have from Corollary 2.1 that

$\sup\{\tau_P(P) \wedge \tau_P(Q) : x \in P, y \in Q, P \cap Q = \phi\}$

$= \sup\{\tau_P(P) \wedge \tau_P(Q) : x \in P \subseteq U, y \in Q \subseteq V, U \cap V = \phi\}$

$= \sup_{U \cap V = \phi} \left\{ \sup_{x \in P \subseteq U} \tau_P(P) \wedge \sup_{y \in Q \subseteq V} \tau_P(Q) \right\} = \sup_{U \cap V = \phi} \left\{ N_x^P(U) \wedge N_y^P(V) \right\}$

$\geq \inf_{x \neq y} \sup_{U \cap V = \phi} \left\{ N_x^P(U) \wedge N_y^P(V) \right\} = T_2^P(X,\tau) = t > \lambda$, i.e.,

there exist P_y, Q_y such that $x \in P_y, y \in Q_y, P_y \cap Q_y = \phi$ and $\tau_P(P_y) > \lambda, \tau_P(Q_y) > \lambda$. Set $\text{ß}(Q_y) = \tau_P(Q_y)$ for $y \in B$. Since $[\text{ß} \subseteq \tau_P] = 1$, we have

$[K_P(\text{ß}, B)] = [K(\text{ß}, B)] = \inf_{y \in B} \sup_{y \in C} \text{ß}(C) \geq \inf_{y \in B} \text{ß}(Q_y) = \inf_{y \in B} \tau_P(Q_y) \geq \lambda$.

On the other hand, Since $T_2^P(X,\tau) \underset{\bullet}{\wedge} (\Gamma_P(A) \wedge \Gamma_P(B)) > 0$, then $1 - t < \Gamma_P(A) \wedge \Gamma_P(B) \leq \Gamma_P(A)$.

Therefore, for any $\lambda \in (1 - \Gamma_P(A), t)$, it holds that

$1-\lambda < \Gamma_P(A) \leq 1-[K_P(ß, B)] + \sup_{\wp \leq ß} \left\{ K(\wp, B)] \underset{\bullet}{\wedge} FF(\wp) \right\}$
$\leq 1-\lambda + \sup_{\wp \leq ß} \left\{ K(\wp, B)] \underset{\bullet}{\wedge} FF(\wp) \right\}$, i.e.,
$\sup_{\wp \leq ß} \left\{ K(\wp, B)] \underset{\bullet}{\wedge} FF(\wp) \right\} > 0$ and there exists $\wp \leq ß$ such that $K(\wp, B) + FF(\wp) - 1 > 0$, i.e., $1 - FF(\wp) < K(\wp, B)$. Then, $\inf \{\theta : F(\wp_\theta)\} < K(\wp, B)$. Now, there exists θ_1 such that $\theta_1 < K(\wp, B)$ and
$F(\wp_{\theta_1})$. Since $\wp \leq ß$, we may write $\wp_{\theta_1} = \{Q_{y_1}, ..., Q_{y_n}\}$. We put $U_x = \{P_{y_1} \cap ... \cap P_{y_n}\}$, $V_x = \{Q_{y_1} \cap ... \cap Q_{y_n}\}$ and have $V_x \supseteq B, U_x \cap V_x = \phi$, $\tau_P(U_x) \geq \tau_P(P_{y_1}) \wedge ... \wedge \tau_P(P_{y_n}) > \lambda$ because (X, τ) is fuzzifying p-topological space. Also, $\tau_P(V_x) \geq \tau_P(Q_{y_1}) \wedge ... \wedge \tau_P(Q_{y_n}) > \lambda$. In fact, $\inf_{y \in B} \sup_{y \in D} \wp(D) = K(\wp, B) > \theta_1$, and for any $y \in B$, there exists D such that $y \in D$ and $\wp(D) > \theta_1, D \in \wp_{\theta_1}$. Similarly, if $\lambda \in (1 - [\Gamma_P(A) \wedge \Gamma_P(B)], t)$, then we can find $x_1, ...x_m \in A$ with $U_\circ = U_{x_1} \cup ... \cup U_{x_m} \supseteq A$. By putting $V_\circ = V_{x_1} \cap ... \cap V_{x_m}$ we obtain $V_\circ \supseteq B, U_\circ \cap V_\circ = \phi$ and
$(\exists U)(\exists V)((U \in \tau_P) \wedge (V \in \tau_P) \wedge (A \subseteq U) \wedge (B \subseteq V) \wedge (U \cap V = \phi)) \geq \tau_P(U_\circ) \wedge \tau_P(V_\circ)$
$\geq \min_{i=1,...,n} \tau_P(U_{x_i}) \wedge \min_{i=1,...,n} \tau_P(V_{x_i}) > \lambda$. Finally, we let $\lambda \to t$ and complete the proof.

(2) Assume $\models^{ws} T_2^P(X, \tau) \underset{\bullet}{\wedge} \Gamma_P(A)$. For any $x \in X - A$ we have from (1)
$\sup_{x \in U \subseteq X - A} \tau_P(U) \geq \sup\{\tau_P(U) \wedge \tau_P(V) : x \in U, A \subseteq V, U \cap V = \phi\} \geq [T_2^P(X, \tau)]$. From Corollary 2.1. we obtain,
$\digamma_P(A) = \inf_{x \in X - A} N_x^P(X - A) = \inf_{x \in X - A} \sup_{x \in U \subseteq X - A} \tau_P(U) \geq [T_2^P(X, \tau)]$. □

Definition 18. *Let (X, τ) and (Y, σ) be two fuzzifying topological spaces. A unary fuzzy predicate $Q_P \in \Im(Y^X)$, called fuzzifying pre-closedness, is given as follows:*

$Q_P(f) := \forall B(B \in \digamma_P^X \to f^{-1}(B) \in \digamma_P^Y)$, where $\digamma_P^X$ and $\digamma_P^Y$ are the fuzzy families of τ, σ-pre-closed in X and Y respectively.

Theorem 11. *Let (X, τ) a fuzzifying topological space, (Y, σ) be an p-fuzzifying topological space and $f \in Y^X$.*
Then $\models \Gamma_P(X, \tau) \underset{\bullet}{\wedge} T_2^P(Y, \sigma) \underset{\bullet}{\wedge} I_P(f) \longrightarrow Q_P(f)$.

Proof. For any $A \subseteq X$, we have the following:
(i) From Theorem 5.1 we have $[\Gamma_P(X, \tau) \underset{\bullet}{\wedge} \digamma_P^X(A)] \leq \Gamma_P(A)$;
(ii)
$$I_P(f_{\backslash A}) = \inf_{U \in P(Y)} \min\left(1, 1 - \sigma_P(U) + \tau_P/_A((f_{\backslash A})^{-1}(U))\right)$$
$$= \inf_{U \in P(Y)} \min\left(1, 1 - \sigma_P(U) + \tau_P/_A(A \cap f^{-1}(U))\right)$$
$$= \inf_{U \in P(Y)} \min\left(1, 1 - \sigma_P(U) + \sup_{A \cap f^{-1}(U) = B \cap A} \tau_P(B)\right)$$
$$\geq \inf_{U \in P(Y)} \min\left(1, 1 - \sigma_P(U) + \tau_P(f^{-1}(U))\right) = I_P(f).$$

(iii) From Theorem 5.3, we have $[\Gamma_P(A) \underset{\bullet}{\wedge} I_P(f_{\backslash A})] \leq \Gamma_P(f(A))$.

(iv) From Theorem 5.4 (2) we have $T_2^P(Y,\sigma) \underset{\bullet}{\wedge} \Gamma_P(f(A)) \models^{ws} T_2^P(Y,\sigma) \longrightarrow f(A) \in \digamma_P^Y$, which implies $\models T_2^P(Y,\sigma) \underset{\bullet}{\wedge} \Gamma_P(f(A)) \longrightarrow f(A) \in \digamma_P^Y$. By combining (i)-(iv) we have

$[\Gamma_P(X,\tau) \underset{\bullet}{\wedge} T_2^P(Y,\sigma) \underset{\bullet}{\wedge} I_P(f)]$
$\leq [(\digamma_P^X(A) \to \Gamma_P(A)) \underset{\bullet}{\wedge} I_P(f_{\backslash A}) \underset{\bullet}{\wedge} T_2^P(Y,\sigma)]$
$\leq [(\digamma_P^X(A) \to (\Gamma_P(A)) \underset{\bullet}{\wedge} I_P(f_{\backslash A}))) \underset{\bullet}{\wedge} T_2^P(Y,\sigma)]$
$\leq [\digamma_P^X(A) \to \Gamma_P(f(A)) \underset{\bullet}{\wedge} T_2^P(Y,\sigma)]$
$\leq [\digamma_P^X(A) \to \digamma_P^Y(f(A))]$. Therefore
$[\Gamma_P(X,\tau) \underset{\bullet}{\wedge} T_2^P(X,\tau) \underset{\bullet}{\wedge} I_P(f)] \leq [\digamma_P^X(A) \to \digamma_P^Y(f(A))]$
$\leq \inf_{A \subseteq X}(\,[\digamma_P^X(A) \to \digamma_P^Y(f(A))]) = Q_P(f)$. □

References

[1] K. M. Abd El-Hakeim, F. M. Zeyada and O. R. Sayed, Pre-continuity and D(c, P)-continuity in fuzzifying topology, *Fuzzy Sets and Systems*, **119** (2001), 459-471.

[2] K. M. Abd El-Hakeim, F. M. Zeyada and O. R. Sayed, Pre-separation axioms in fuzzifying topology, *Fuzzy Systems and Mathematics*, **17** (1) 2003, 29-37.

[3] R. H. Atia, S. N. El-Deep, I. A. Hasanein, A note on strong compactness and S-closedness, *Math. Vesnik,* **6** (19) (34) (1982) 23-28.

[4] A. S. Masshour, M. E. Abd El-Monsef, I. A. Hasanein and T. Noiri, Strongly compact spaces, *Delta J. Sci.*, **8** (1) (1984), 30-46.

[5] S. Nanda, Strongly compact fuzzy topological spaces, *Fuzzy Sets and Systems*, **42** (1991), 259-262.

[6] J. H. Park and B. H. Park, Fuzzy pre-irresolute mappings, *Pusan-Kyongnam Math. J.*, **10** (1995), 303-312.

[7] I. L. Reilly and M. K. Vamanamurthy, On α-continuity in topological spaces, *Acta Math. Hung.*, **45** (1-2)(1985), 27-32.

[8] M. S. Ying, A new approach for fuzzy topology (I), *Fuzzy Sets and Systems*, **39** (1991), 303-321.

[9] M. S. Ying, A new approach for fuzzy topology (II), *Fuzzy Sets and Systems*, **47** (1992), 221-23.

[10] M. S. Ying, A new approach for fuzzy topology (III), *Fuzzy Sets and Systems*,**55** (1993), 193-207.

[11] M. S. Ying,, Compactness in fuzzifying topology, *Fuzzy Sets and Systems*, **55** (1993), 79-92.

[12] A. M. Zahran, Strongly compact and P-closed fuzzy topological spaces, *J. Fuzzy Math.*, **3**(1) (1995), 97-102.

Received 29/3/2004
Revised 5/9/2006